HONOM 2024

10th International Conference on

High-Order Nonlinear numerical Methods for evolutionary PDEs: theory and applications

Book of Abstracts

Main organizers: Elena Gaburro & Maria Kazolea

"*Behind every result is a new challenge*"

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Welcome to HONOM 2024!

We are very pleased to welcome you to HONOM 2024, *the 10th International Conference on High-Order NOnlinear numerical Methods for evolutionary PDEs: theory and applications*, held at the KAM Conference Center hosted in the historical Megalo Arsenali, in front of the sea in the picturesque town of Chania, in the western part of Crete island in Greece.

HONOM 2024 builds on the success of its previous editions. Everything started in Trento (Italy) in 2005: the event was promoted by R. Abgrall, M. Dumbser, C.-D. Munz and E.F. Toro and it was reproposed there in 2007, 2009, 2011 and 2015. Then, HONOM has been organized in Bordeaux (France) in 2013, Stuttgart (Germany) in 2017, Madrid (Spain) in 2019 and Braha (Portugal) in 2022.
This year we have received 93 abstracts, which have been carefully reviewed by the members of the scientific and organizing committee. There will be 63 talks and 14 poster presentations, given by international researchers coming from *Canada, China, Czech Republic, France, Germany, Greece, Italy, Netherlands, Portugal, Slovakia, Spain, Sweden, Switzerland, the United Kingdom, and the United States of America.* Unfortunately our colleagues from *Russia*, whose abstracts were accepted, could not come.

The aim of this conference is to present new research on advanced mathematical models and advanced numerical algorithms for the robust and effective solution of evolutionary PDEs of interest in a wide range of physically relevant situations as computational fluid and solid mechanics, multiphase flows, oceanography, plasma physics, material science, mathematical biology and computational astrophysics. We will treat the design of novel algorithms, the analysis and applications of non-linear schemes of accuracy greater than two, which follows the finite difference, finite volume, finite element or residual distribution approaches; also, we will discuss structure preserving numerical methods and PDE models; moreover, we will enhance the state of the art on mesh generation, motion and adaptation taking into account their strict connection with the development of effective numerical methods.

We would really like to thank the scientific committee, the local organizers, our sponsors and in particular *all the participants* for coming to HONOM 2024.
We wish you a very pleasant stay in Crete and many enriching scientific and personal interactions during the conference!

Elena & Maria

Committees and Invited Speakers

Scientific committee:

Rémi Abgrall *(University of Zurich, Switzerland)*

Michael Dumbser *(University of Trento, Italy)*

Claus-Dieter Munz *(University of Stuttgart, Germany)*

Eleuterio F. Toro *(University of Trento, Italy)*

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Co-chair: Maria Kazolea *(Inria Bordeaux, France)*

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Local organizers:

Anargiros Delis *(Technical University of Crete, Chania, Greece)*

Anastasios Sifalakis *(Technical University of Crete, Chania, Greece)*

Invited keynote speakers:

Paola F. Antonietti *(Politecnico di Milano, Italy)*

Florent Renac *(Onera, France)*

Christian Rohde *(University of Stuttgart, Germany)*

Matteo Semplice *(Università dell'Insubria, Italy)*

Panagiotis Tsoutsanis *(Cranfield University, United Kingdom)*

Karen Veroy-Grepl *(Eindhoven University, The Netherlands)*

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Conference Format and Conference Venues

Conference format:

Keynote talks are plenary, held in the KAM conference room and last **50 minutes** included questions.

Contributed talks may be plenary (KAM conference room), or divided into 2 parallel sections (KAM and MIKIS conference rooms).
They should last **25 minutes** included questions.

Contributed poster will be hung in the KAM center and presented on Tuesday evening.

Registration and Welcome Reception

Chania Sailing Club Neorio Moro

Sunday 8 of September 2024

17:30 – 20:00: **Registration**

19:15 – 22:00: **Welcome reception**

Address: Chania Old Town Marina, G29F+Q8 Chania, Greece

Main Conference Venue: keynote talks, contributed talks and posters

The two buildings are on the two sides of the same square; consider 5 minutes between them.

KAM conference center

Address: Chania Old Town Marina, G299+CW Chania, Greece

MIKIS Theodorakis Theatre

Address: Chania Old Town Marina, G29C+75 Chania, Greece

Conference Dinner: Nykterida Restaurant Bar

Thursday 12 of September 2024

19:15 – 19:20: Bus departure from the opposite side w.r.t the Bank of Chania, Chania city center.
Google maps link: here

20:00 – 24:00: Conference dinner

Address (Nykterida): G38G+PJ Kounoupidiana, Greece (25 minutes by bus from the main event venue)

Monday 9 of September 2024

08:00 – 09:15 **REGISTRATION**

09:15 – 09:25 **OPENING**

Chair: M. Semplice

09:25 **Panagiotis Tsoutsanis**
ADDAptive Numerical Framework for iLES of Compressible Flows

10:15 **Elena Gaburro**
A primitive-conservative ADER-DG method for multiphase flows on polygonal meshes

10:40 – 11:25 **COFFEE BREAK**

KAM

Chair: J. Nordström

11:25 **Thomas Izgin**
High-Order Positivity-Preserving Methods for Hyperbolic Balance Laws

11:50 **Philippe Hoch**
Arbitrary high-order [...] composite FV schemes with induced physically admissible reconstruction

12:15 **Irene Gómez-Bueno**
Preserving non-moving steady states for Euler [...] with gravitational forces and the Ripa model

MIKIS

Chair: P.-O. Persson

Anna Schwarz
Entropy stable shock capturing for high-order discontinuous Galerkin schemes on moving meshes

Vladimir Tomov
Slip Wall BC in Curved Domains for FE ALE Hydrodynamics

Patrick Kopper
A Curvilinear Euler–Lagrange Code on Unstructured Moving Meshes

12:40 – 14:40 **LUNCH**

KAM

Chair: V. Perrier

14:40 **Francesco Carlo Massa**
Hybrid High-Order methods with hybrid pressure and improved turbulence modelling capabilities

15:05 **Emanuele Carnevali**
Efficient Compressible Turbulent Flow Simulations: The Impact of Entropy Projection and [...]

15:30 **Ricardo Costa**
Very high-order accurate FV for the streamfunction-vorticity formulation of incompressible [...]

MIKIS

Chair: A. Kurganov

Ernesto Pimentel-García
In-cell Discontinuous Reconstruction path-conservative methods for nonconservative hyperbolic [...]

Julie Patela
Arbitrary-order finite volume schemes preserving positivity for diffusion

Nikita Afanasev
Towards a High-Order Conservative-Characteristic CABARET Scheme

15:55 – 16:35 **COFFEE BREAK**

Chair: C.-D. Munz

16:35 **Per-Olof Persson**
Half-Closed Discontinuous Galerkin Discretisations

17:00 **Paola Antonietti**
High-order discontinuous polytopal methods for modeling neurodegeneration

Tuesday 10 of September 2024

Chair: C. Klingenberg

09:00 **Dinshaw Balsara**
General Purpose Alternative Finite Difference WENO for Conservative and Non-Conservative [...]

09:50 **Andrés M. Rueda-Ramírez**
A Robust Entropy-Stable Discontinuous Galerkin Scheme for the Multi-Ion MHD System

10:15 – 11:00 COFFEE BREAK

KAM

Chair: A. Delis

11:00 **Luca Alberti**
On the high-order implementation of hybrid RANS/LES models for flapping foils

11:25 **Alessandro Colombo**
On the implementation of a wall model for implicit LES in an entropy-stable DG solver

11:50 **Satyvir Singh**
DG for continuum-rarefied gas flows over aerospike blunt body based on regularized 13-moment model

12:15 **Cristian Brutto**
A semi-implicit finite volume scheme for the simulation of floating objects

MIKIS

Chair: V Perrier

Francesco Fambri
Structure Preserving Hybrid Finite Element - Finite Volume for MHD

Enrico Zampa
Compatible FE discretization of time-dependent magnetic advection-diffusion [...] to MHD

José Castillo
Energy Preserving High Order Mimetic Methods For Hamiltonian Systems

Tarik Dzanic
Towards full Boltzmann simulations of complex fluid flows via high-order discretely-conservative [...]

12:40 – 14:40 LUNCH

KAM

Chair: S. Busto

14:40 **Axelle Drouard**
Semi-implicit numerical scheme for hyperbolic problems

15:05 **Katarína Lacková**
High-resolution compact semi-implicit level set methods for the advection equation

15:30 **Peter Frolkovic**
Compact implicit numerical schemes for nonlinear hyperbolic systems

MIKIS

Chair: V. Tomov

Catherine Mavriplis
Pushing the Geometrical Capabilities of High Order Galerkin Spectral Element Methods

Jens Keim
An Efficient Discontinuous Galerkin Spectral Element Implementation on Heterogeneous Grids

Ketan Mittal
Scalable Interpolation at Arbitrary Points in High-Order Volume and Surface Meshes on GPUs

15:55 – 16:35 COFFEE BREAK

Chair: S. Chiocchetti

16:35 **Alexander Kurganov**
A Well-Balanced Fifth-Order A-WENO Scheme Based on Flux Globalization

17:00 **Matteo Semplice**
QUINPI: going implicit for nonlinear hyperbolic equations

18:00 – 21:00 POSTER PARTY

Wednesday 11 of September 2024

Chair: W. Barsukow

09:00 **Vincent Perrier**
How to preserve a divergence or a curl constraint in a hyperbolic system with the DG method

09:25 **Davide Torlo**
Divergence–free preserving schemes: how to fix stabilization terms in continuous Galerkin

09:50 **François Vilar**
Monolithic local subcell DG/FV convex property preserving scheme: is entropy stability really needed?

10:15 **Alina Chertock**
Adaptive High-Order A-WENO Schemes Based on a New Local Smoothness Indicator

10:40 – 11:25 COFFEE BREAK

KAM

Chair: F. Fambri

11:25 **Davide Ferrari**
A unified SHTC multiphase model of continuum mechanics

11:50 **Daniel Regener Roig**
Entropy-stable DG solution of the multicomponent Euler [...] with entropy balance enforcement

12:15 **Susana Serna**
High-Order Shock-Capturing Schemes for Non-Convex Special Relativistic Hydrodynamics

12:40 **Juan Cheng**
High order conservative numerical schemes for three-temperature radiation hydrodynamics

MIKIS

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Celia Caballero-Cárdenas
Semi-implicit finite volume schemes for systems of shallow flows: preserving every steady state

A. González del Pino
2nd and 3rd order FV for the 2D SWE in spherical coordinates with non-constant Coriolis [...]

Gaspar Machado
R-Block structural schemes for ordinary differential equations

Alexis Tardieu
A class of high order ADER-DG schemes for [...] nonlinear advection-diffusion equation

FREE AFTERNOON

Thursday 12 of September 2024

09:00 **Karen Veroy-Grepl**
Challenges for Physics-Based Model Order Reduction in Data Assimilation

09:50 **Georgios Kokkinakis**
Troubled-cell detection for high-order methods on unstructured meshes by convolution neural networks

Chair: D. Torlo

10:15 – 11:00 COFFEE BREAK

11:00 **Christian Klingenberg**
On a semi-discrete Active Flux method for multi-dimensional conservation laws

11:25 **Lisa Lechner**
A two-dimensional Active Flux method of arbitrarily high order

11:50 **Junming Duan**
On limiting for the Active Flux methods for hyperbolic conservation laws

12:15 **Wasilij Barsukow**
Stability of extensions of Active Flux

Chair: J. Cheng

12:40 – 14:40 LUNCH

14:40 **Jan Nordström**
An Energy Stable Nonlinear Incompressible Multi-Phase Flow Formulation

15:05 **Firas Dhaouadi**
A first-order hyperbolic reformulation of the Cahn-Hilliard equation

15:30 **Saray Busto**
A semi-implicit hybrid finite volume/finite element method for continuum mechanics

Chair: F. Renac

15:55 – 16:35 COFFEE BREAK

16:35 **Simone Chiocchetti**
Hyperbolic viscous flow using quaternion fields

17:00 **Christian Rohde**
Numerics for compressible liquid-vapour flow: sharp-interface and diffuse-interface models

Chair: F. Dhaouadi

19:15 – 23:30 SOCIAL DINNER

Nykterida Restaurant Bar
EO Aerodromiou Soudas 3, Kounoupidiana 73100, (G38G+PJ) Greece,
25 minutes by bus from the main event venue.
Note: buses to go to the restaurant will leave at 19:15–19:20,
from the city center of Chania, opposite side of the street w.r.t the *Bank of Chania*
Google maps link: here

Friday 13 of September 2024

09:25 **Florent Renac**
Positivity preserving time implicit DGSEM for hyperbolic conservation laws

Chair: P. Tsoutsanis

10:15 – 11:00 COFFEE BREAK

11:00 **Lilia Krivodonova**
Limiters for the Discontinuous Galerkin Method on Quadrilateral Meshes

11:25 **Joshua Vedral**
Strongly consistent low-dissipation WENO schemes for finite elements

11:50 **Malte Wegener**
P-Anisotropic H-Isotropic adaptive discontinuous Galerkin methods for turbulent flows

12:15 **Claus-Dieter Munz**
An h-p Adaptive Strategy for Discontinuous Galerkin Schemes

Chair: F. Vilar

12:40 – 14:00 CONCLUSIVE APERO

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Towards a High-Order Conservative-Characteristic CABARET Scheme

N. Afanasev[†] **and V. Goloviznin**[‡]

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[‡] Department of Computational Methods, Lomonosov Moscow State University (gol@ibrae.ac.ru)

Keywords: conservative-characteristic methods, hyperbolic PDEs, Riemann invariants.

ABSTRACT

CABARET scheme is an explicit conservative-characteristic second-order numerical method for hyperbolic PDEs. It uses a finite volume approach to approximate conservation laws in the cells of a mesh, and a characteristic approach to approximate fluxes (using Riemann invariants) on the next time layer. The scheme has a computational stencil of minimal size (one space-time cell) and it is time-reversible (non-dissipative) if monotonization procedures are not used.

CABARET's characteristic phase uses the extrapolation of Riemann invariants along the linearized characteristics, which makes the process of increasing the order of approximation rather difficult. One must take into account the approximation of the characteristic itself, the approximation of Riemann invariants on it and the high-order extrapolation procedure. The finite volume part of the scheme also needs to be corrected.

In this work we discuss several approaches to upgrading the CABARET's order of approximation. The first way involves the classical expansion of the scheme's finite volume part stencil and a thorough analysis of how it affects the characterisitic phase. The second approach uses additional degrees of freedom inside computational cells. Some one-dimensional tests for linear and non-linear problems are presented, including acoustic and Riemann problems.

References

[1] S. Karabasov and V. Goloviznin "Compact Accurately Boundary-Adjusting High-REsolution Technique for Fluid Dynamics", *Journal of Computational Physics*, 228(19), pp. 7426–7451, 2009.

[2] N. Afanasiev and V. Goloviznin, "A Locally Implicit Time-Reversible Sonic Point Processing Algorithm for One-Dimensional Shallow-Water Equations", *Journal of Computational Physics*, 434, p. 110220, 2021.

[3] N. Afanasiev, V. Goloviznin and A. Solovjev "CABARET Scheme with Improved Dispersion Properties for Systems of Linear Hyperbolic-Type Differential Equations", *Numerical Methods and Programming*, 22, pp. 67–76, 2021.

On the high-order implementation of hybrid RANS/LES models for flapping foils

L. Alberti[†*], **E. Carnevali**[†], **A. Colombo**[‡], **A. Crivellini**[†] **and G. Noventa**[§]

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Keywords: Discontinuous Galerkin, DDES, IDDES, flapping foils, non-inertial frame, adaptive time step.

ABSTRACT

In the present work we investigate the performance of several hybrid RANS/LES models in a high-order discretization framework. Many hybrid models are indeed tailored around finite volume methods, therefore it is of particular interest to examine their performance within the context of high-order schemes. The numerical study is carried out using an incompressible modal discontinuous Galerkin solver. The assessment includes multiple turbulence models, e.g. the DDES of Spalart et al. [1] and the IDDES of Shur et al. [2]. Cutting-edge shielding functions and subgrid length scales are tested in an attempt to find the most suitable ones for simulating external highly unsteady flows. The effect of low-Reynolds corrections and the use of underlying algebraic transition RANS models is also taken into consideration. The analysis involves the simulation of three-dimensional flapping hydrofoils under various kinematic conditions. To avoid remeshing or deforming the computational grid at each time step we introduce a non-inertial reference frame integral with the foil. The implementation details can be found in [3]. Since for a fixed time step size one would be constrained by the very fast dynamics occurring at half of the period in the sinusoidal motion of the foil, we utilize an implicitly adaptive time stepping scheme. The algorithm is based on the one designed in [4] and also enables to adapt the tolerance of the linear solver at each time step.

References

[1] P. R. Spalart, S. Deck, M. L. Shur, K. D. Squires, M. K .Strelets and A. K. Travin, A new version of detached-eddy simulation, resistant to ambiguous grid densities. *Theoretical and computational fluid dynamics*, Vol. 20, pp. 181-195, 2006.

[2] M. L. Shur, P. R. Spalart, M. K., Strelets and A. K. Travin, A hybrid RANS-LES approach with delayed-DES and wall-modelled LES capabilities. *International journal of heat and fluid flow*, Vol. 29(6), pp. 1638-1649, 2008.

[3] L. Alberti, E. Carnevali and A. Crivellini, Assessment of a RANS Transition Model with Flapping Foils at Moderate Reynolds Numbers, *Fluids*, Vol. 8.1, pp. 23, 2023.

[4] G. Noventa, F. Massa, S. Rebay, F. Bassi and A. Ghidoni, Robustness and efficiency of an implicit time-adaptive discontinuous Galerkin solver for unsteady flows, *Computers & Fluids*, Vol. 204, p. 104529, 2020.

High-order discontinuous polytopal methods for modeling neurodegeneration

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[†] Laboratory for Modeling and Scientific Computing (MOX), Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, ITALY
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Keywords: polytopal discontinuous high-order methods, neurodegenerative diseases.

ABSTRACT

Neurodegenerative diseases (NDs) are complex disorders that primarily affect neurons in the brain and nervous system, leading to progressive deterioration and loss of function over time. A common pathological hallmark among various NDs is the accumulation of disease-specific misfolded proteins, such as amyloid-beta and tau in Alzheimer's disease, and alpha-synuclein in Parkinson's disease. In this talk, we discuss the mathematical and numerical modeling of misfolded protein dynamics in neurodegenerative diseases, employing mathematical models of increasing complexity. To tame complexity, we propose and analyze high-order discontinuous Galerkin methods on polytopal grids (PolyDG) for numerical discretization. We present numerical simulations using patient-specific brain geometries reconstructed from clinical data. In the second part of the talk, we discuss the computational modeling of mechanisms for waste removal (clearance) from the brain, which plays a crucial role in the onset and progression of NDs. We present and analyze our numerical approach, along with patient-specific simulations.

General Purpose Alternative Finite Difference WENO (AFD-WENO) for Conservative Systems and Systems with Non-Conservative Products

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† Physics and ACMS, University of Notre Dame, USA (dinshaw.d.balsara.1@nd.edu)

Keywords: high order numerical methods, conservative and non-conservative systems

ABSTRACT

In their landmark sequence of papers (Shu and Osher 1988, 1989) the authors presented two highly efficient finite difference WENO schemes. The latter finite difference WENO scheme (Shu and Osher 1989) has become wildly successful and garnered thousands of citations. We call that the FD-WENO scheme. However, in Shu and Osher (1988) they also presented an alternative finite difference WENO (AFD-WENO) scheme which was slower to catch on. We explain why that scheme was slower to catch on – it is because all ingredients that are needed to make a production code out of AFD-WENO were not available at that time. Besides, the scheme was not easy to understand at the time of its initial presentation. We demystify the AFD-WENO algorithm in this talk. In this talk we explain why the AFD-WENO scheme, nevertheless, had several significant advantages, if it could be developed into an automated algorithm for production codes. This talk is devoted to developing AFD-WENO into a simple algorithm that is easily explained to others and also easily implemented in production codes. To reach that goal, we had to make several algorithmic innovations which we explain here. The original FD-WENO schemes were also only viable for conservation laws. But the field has moved on and it is very normal for scientists and engineers to discover hyperbolic PDE systems that have non-conservative products, often with stiff source terms. To accommodate such PDE systems, we present the first of its kind AFD-WENO scheme that can retain strict conservation when the PDE is conservative, and yet, accommodate non-conservative products. This vastly expands the class of PDEs that can be treated with AFD-WENO schemes. Several examples are demonstrated in this talk. Very recent advances in PCP and well-balancing of these schemes will also be presented.

Stability of extensions of Active Flux

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Keywords: Conservation laws, Active Flux, high order.

ABSTRACT

Active Flux is a recently developed numerical method for hyperbolic conservation laws ([5]). We focus here on the one-dimensional case. The degrees of freedom are cell averages and point values at cell boundaries, which are shared by the adjacent cells. This gives rise to a globally continuous reconstruction that is locally a parabola, such that the original method is third-order accurate.

The update of the averages is immediate because the pointwise degrees of freedom can be immediately used as quadrature points for the fluxes along cell boundaries (hence the name of the method). Several suggestions have been made about how to update the point values, and they will be reviewed during this talk. Traditionally, the reconstruction serves as initial data for an (approximate) evolution operator that immediately updates the point values to the new time level (e.g. [4]). More recently, semi-discrete approaches have also been considered, with finite differences to approximate the spatial derivatives and standard (Runge-Kutta) methods to update in time (method of lines) [1]. At this point, Active Flux can be understood as a hybrid finite-volume/finite-element method.

During the talk, several new suggestions for the point value update will be analyzed and compared. They all include upwinding, and a particular focus will be placed on a stability analysis of the proposed extensions.

References

[1] R. Abgrall and W. Barsukow. Extensions of Active Flux to arbitrary order of accuracy. *ESAIM: Mathematical Modelling and Numerical Analysis*, 57(2):991–1027, 2023.

[2] R. Abgrall and W. Barsukow. A hybrid finite element–finite volume method for conservation laws. *Applied Mathematics and Computation*, 447:127846, 2023.

[3] R. Abgrall, W. Barsukow, and C. Klingenberg. A semi-discrete active flux method for the euler equations on cartesian grids. *subm.*, 2023.

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[5] T. A. Eymann and P. L. Roe. Multidimensional active flux schemes. In *21st AIAA computational fluid dynamics conference*, 2013.

Virtual Element Complexes of general order and application to MagnetoHydroDynamics

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Keywords: magnetohydrodynamics, virtual elements, exact discrete complexes, polytopal meshes

ABSTRACT

The Virtual Element Method (VEM) is a technology introduced in 2013 for the discretization of partial differential equations that follows a similar paradigm to classical Finite Elements, but with important differences. By avoiding the explicit integration of the discrete shape functions and introducing an innovative construction of the stiffness matrixes, the VEM acquires interesting properties and advantages with respect to more standard methods. For instance, the VEM easily allows for general polygonal/polyhedral meshes, even non-conforming and with non-convex elements.

The present talk will take the steps from a classical model in non-stationary magnetohydrodynamics and focus on presenting an $H^1 - H_{\text{curl}} - H_{\text{div}} - L^2$ discrete exact complex of general "polynomial" order [1, 2, 3], based on VEM and applicable to general polyhedral grids. This complex plays a critical role in the approximation of many differential problems, for instance in porous media flows and electromagnetism. The presentation will detail the two-dimensional case, including also a more complicated but efficient set of spaces, and afterwards discuss the extension to the three-dimensional case. The talk will close with some numerical tests, showcasing the application of the aforementioned complex to non-stationary magnetohydrodynamics [4].

References

[1] L. Beirão da Veiga, F. Brezzi, F. Dassi, L.D. Marini, L. D., A. Russo. "Virtual Element approximation of 2D magnetostatic problems", CMAME, 327:173-195 (2017).

[2] L. Beirão da Veiga, F. Brezzi, F. Dassi, L.D. Marini, L. D., A. Russo. "A family of three-dimensional virtual elements with applications to magnetostatics", SINUM 56:2940-2962 (2018).

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[4] L. Beirão da Veiga, F. Dassi, G. Manzini, L. Mascotto. "The virtual element method for the 3D resistive magnetohydrodynamic model", M3AS 33:643-686 (2023)

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A semi-implicit finite volume scheme for the simulation of floating objects

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Keywords: Fluid-structure interaction, Semi-implicit finite-volume scheme, Subgrid, Nested Newton, Shallow water

ABSTRACT

In this talk, a novel semi-implicit finite volume scheme (SIFSI) is presented for the coupled solution of the water flow and the movement of one or more floating structures. The model is based on the hydrostatic pressure assumption and the shallow water equations. The coupling is achieved via a nonlinear volume function in the mass conservation equation that depends on the coordinates of the floating structures. The resulting mildly nonlinear pressure system is solved using a nested Newton method. The accuracy of the volume computation is improved by using a subgrid.
The horizontal motion, which is important for maneuvers, presented a numerical instability that was not observed in the test cases with only the vertical motion. This instability is related to the relative movement between the ship and the grid. When the ship enters a new cell, the pressure sharply increases and decreases at the bow, leading to oscillations that can create an unphysical void below the vessel. Several ideas were implemented to reduce the oscillations. All these modifications were effective at controlling the oscillations, and simulations with horizontal motion are now much more stable.
The model is able to treat fluid-structure interaction in the context of geophysical free surface flows in an efficient and flexible way, and the employed nested Newton method rapidly converges to a solution.

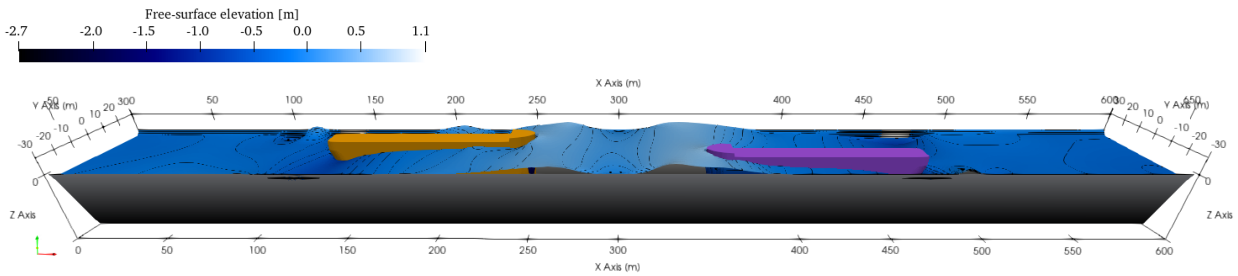

Figure 1: Simulation of the encounter of two ships.

References

[1] C. Brutto and M. Dumbser, "A semi-implicit finite volume scheme for a simplified hydrostatic model for fluid-structure interaction", *International Journal for Numerical Methods in Fluids*, 95(1), pp. 107–142, 2023.

This project is part of a collaboration between the University of Trento and Bundesanstalt für Wasserbau.

A semi-implicit hybrid finite volume/finite element method for continuum mechanics

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Keywords: GPR model for continuum mechanics, semi-implicit structure preserving schemes, hybrid finite volume/finite element methods, unstructured staggered grids.

ABSTRACT

We propose a novel semi-implicit hybrid finite volume (FV) / finite element method (FE) for the solution of continuum mechanics. In particular, we consider the incompressible and weakly compressible formulations of the first order hyperbolic Godunov-Peshkov-Romenski model (GPR) which, at the aid of suitable relaxation source terms, is able to describe continuum mechanics problems going from non-linear elasto-plastic solids to viscous fluids [1]. Following the splitting strategy proposed in [2] we obtain a pressure subsystem of the Poison type and a transport subsystem containing the convective terms and non-conservative products. Then, attending to the different nature of the equations, suitable numerical methods are applied for its solution, namely explicit finite volume schemes for the transport equations and implicit continuous finite elements for the pressure subsystem, [3]. This splitting frees the dependency of the time step restriction on the sound velocity resulting on a well suited methodology for the solution of low Mach number problems. To ease the discretization of complex domains while avoiding stability issues, the methodology is developed employing unstructured staggered grids. The proposed approach is carefully validated on a wide set of benchmarks going from incompressible flows and moderate Mach number test cases to classical problems in solid mechanics.

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Semi-implicit finite volume schemes for systems of shallow flows: preserving every steady state

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Keywords: semi-implicit schemes, fully exactly well-balanced schemes, systems of shallow flows.

ABSTRACT

This work is centered on the design of semi-implicit schemes for one dimensional shallow flows, ensuring preservation of all steady states, not just water-at-rest ones. In order to do so, the authors follow the idea of some previous works (see [1], [2]) which use a combination of splitting and relaxation techniques to avoid nonlinearities related to the pressure terms. The proposed methods outperform standard explicit schemes in the low Froude regime, where celerity is larger than fluid velocity, avoiding the need for many iterations on large time intervals. Finally, some numerical examples are presented to check the good performance of the schemes.

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Efficient Compressible Turbulent Flow Simulations: The Impact of Entropy Projection and Correction on DG ILES

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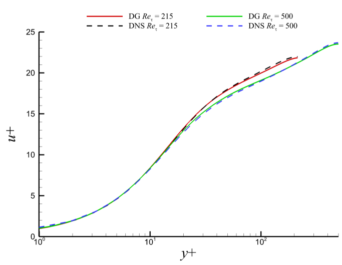
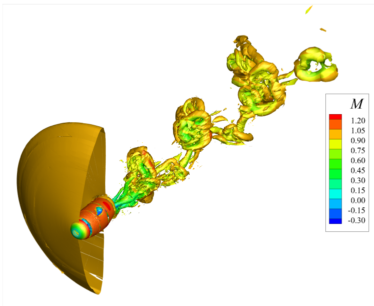
Keywords: Discontinuous Galerkin, ILES, compressible turbulent flows, entropy projection, entropy correction.

ABSTRACT

In the current literature under-resolved DG simulations are usually performed at low Mach numbers [1], with the robustness of the approach still being questionable under transonic/supersonic conditions. Here, we report on a standard DG scheme embedded with an entropy projection-correction [2, 3], as an effective and efficient solution for undertaking the demanding conditions of scale-resolving simulations of transonic/supersonic flows. The scale-resolving capability of the method is assessed by drawing comparisons from the DNS results referring to both channel transonic/supersonic flows and flows impinging on a sphere. Concerning the former, we observe that for configurations characterized by $M = 1.5$ and $Re_\tau = 215, 500$ the adopted numerical framework provides highly accurate results: for a DG polynomial order $k = 6$ the mean velocity profiles collapse almost uniformly onto the DNS data, Fig. 1a, while retaining only a fraction of the degrees of freedom of the reference solution. Similar accordance with the reference DNS dataset is also obtained for a sphere at $M = 0.2, 0.95$ and $Re = 1000$, with remarkably accurate results even on a severely under-resolved discretization: both St number and $\overline{C}_D$ are effectively predicted without employing any shock capturing algorithm. In addition, we show that the entropy projection-correction framework provides a robustness improvement to the baseline conservative scheme while maintaining a comparable computational cost.



(a) $< u_1 > /u_\tau$ profile.



(b) $M = 0.95$, $k = 4$.

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Energy Preserving High Order Mimetic Methods For Hamiltonian Systems

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Keywords: high order mimetic methods, energy preserving schemes, Hamiltonian systems.

ABSTRACT

Hamiltonian equations possess a Hamiltonian function that govern the conserved physical property for the system. Obtaining a discretization scheme that satisfies the intrinsic geometric properties of its continuum problem is often a challenge. Spatial schemes that discretely mimic a conservation law are known to result in accurate discretizations of partial differential equations. The mimetic methods of Castillo et al [1, 2, 3] result in high order mimetic operators which, by construction, mimic the extended Gauss divergence theorem. These operators work on staggered spatial grids and produce even orders of accuracy at the boundaries and interiors, without the use of ghost nodes. The high order mimetic operators *DIV* and *GRAD* are discrete approximations of their continuum counterpart vector calculus identities of divergence and gradient. The resulting discretizations are therefore said to mimic the underlying physics. The preservation of the spatio-temporal energy evolution requires a corresponding time integration schemes that is structure preserving, such as the staggered leapfrog scheme. The traditional leapfrog scheme, however, is limited to second order accuracy. In this work, we study the high order composition temporal methods with the mimetic operators to investigate the energy preserving aspects of Hamiltonian systems. Fourth and sixth order spatio-temporal energy preserving scheme shall be presented for both linear and non-linear Hamiltonian systems. Numerical examples that illustrate our findings shall also be presented.

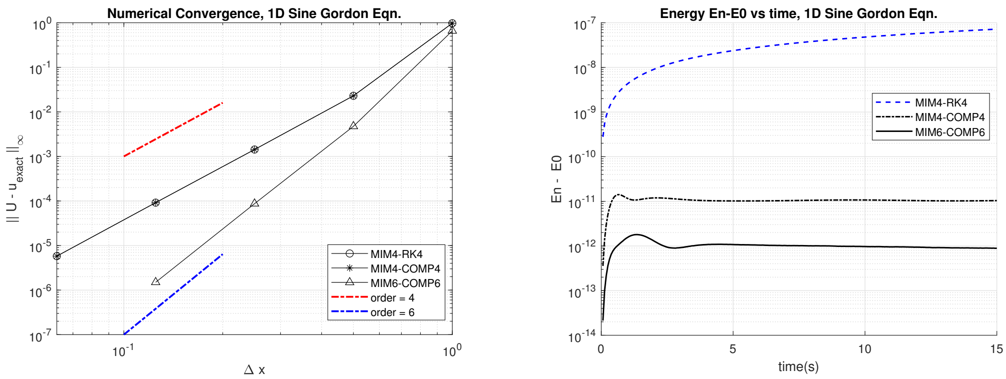

Figure 1: Numerical convergence and energy evolution, 1D Sine Gordon equation

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High order conservative numerical schemes for three-temperature radiation hydrodynamics equations

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Keywords: Three-temperature, Radiation hydrodynamics equations, High order, Conservative, Lagrangian methods.

ABSTRACT

The three-temperature (3-T) radiation hydrodynamics (RH) equations are widely used in modeling various optically thick high-energy-density-physics environments, such as those in astrophysics and inertial confinement fusion. In this talk, we will discuss the methodology to construct high order conservative numerical schemes solving the 3-T RH equations. Specifically, the three new energy variables are defined first, in the form of which the three energy equations of the 3-T RH equations are rewritten. The main advantage of this formulation is that it facilitates the design of a scheme with both conservative property and arbitrary high order accuracy. Based on the WENO reconstruction and the strong stability preserving (SSP) high order time discretizations, we design high order conservative schemes both in space and time. Finally, various numerical tests are given to verify the desired properties of the schemes such as high order accuracy, non-oscillation and conservation.

Adaptive High-Order A-WENO Schemes Based on a New Local Smoothness Indicator

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Keywords: local smoothness indicator, scheme adaption, strong stability preserving Runge-Kutta methods, hyperbolic systems of conservation laws, A-WENO schemes.

ABSTRACT

In this talk, we present new adaptive alternative weighted essentially non-oscillatory (A-WENO) schemes for hyperbolic systems of conservation laws. The new schemes, developed in [1], employ the recently proposed local characteristic decomposition-based central-upwind numerical fluxes, the three-stage third-order strong stability preserving Runge-Kutta time integrator, and the fifth-order WENO-Z interpolation. The adaptive strategy is implemented by applying the limited interpolation only in the parts of the computational domain where the solution is identified as "rough" with the help of a smoothness indicator. We develop and use a new simple and robust local smoothness indicator (LSI), which is applied to the solutions computed at each of the three stages of the ODE solver.

The new LSI and adaptive A-WENO schemes are tested on the Euler equations of gas dynamics. We implement the proposed LSI using pressure, which remains smooth at contact discontinuities. Our goal is to detect other "rough" areas and apply the limited interpolation mainly in the neighborhoods of the shock waves. We demonstrate that the new adaptive schemes are highly accurate, non-oscillatory, and robust. They outperform their fully limited counterparts (the A-WENO schemes with the same numerical fluxes and ODE solver but with the WENO-Z interpolation employed everywhere) while being less computationally expensive.

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Hyperbolic viscous flow using quaternion fields

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Keywords: Hyperbolic equations, Finite Volume methods, Quaternion fields, Stiff sources, Relaxation systems.

ABSTRACT

This talk is concerned with representation issues associated with the numerical solution of a unified mathematical model of continuum mechanics, due to Godunov, Peshkov, and Romenski, which *can describe ideal fluids, viscous fluids and elastoplastic solids as special cases of a general continuum* [1, 2]. The different regimes are characterized solely by the choice of material parameters and the resulting PDE system is of hyperbolic nature, with clearly defined finite wave speeds, in contrast to the standard formulation of viscous fluids via the Navier–Stokes equations.

The description of such a general continuum hinges on the evolution of a matrix-valued field called distortion, which is a generalization of the inverse deformation gradient in solid mechanics. In the fluid regime, this quantity can no longer be recovered as a gradient of displacements and encodes very rich information, in particular due to the different orientations that ideal fluid parcels can be found in. The fine features of the distortion field can be challenging (or outright impossible) to resolve with standard well-tested Finite Volume methods. Degenerate situations are routinely encountered where unphysical states are generated simply as a result of taking a convex combination of two data points.

We show how changing to an alternative representation of the same distortion field, obtained via polar decomposition, can be used to solve such discretization issues.

Instead of the original PDE system, one can instead evolve the rotational and stretch components of the distortion matrix separately, which allows the description of the rotational components through a *quaternion-valued partial differential equation*. We discuss the peculiarities of quaternion PDEs and some of the discretization strategies that they enable. We present numerical examples of high-Reynolds number simulations which could not be carried out with the previous formulation of the model.

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On the implementation of a wall model for implicit Large Eddy Simulation in an entropy-stable discontinuous Galerkin solver

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Keywords: discontinuous Galerkin, entropy-stability, scale-resolving simulations, Wall-Model iLES.

ABSTRACT

Discontinuous Galerkin methods are well suited for the implicit Large Eddy Simulation (iLES) of compressible turbulent flows, where the unfiltered Navier-Stokes equations are solved, and the numerical dissipation acts as a subgrid-scale model dissipating the smallest eddies. While LES may be a viable approach for low and moderate Reynolds number flow problems, its application to high Reynolds number cases is currently computationally prohibitive due to the demanding resolution requirements at walls. In order to mitigate this cost, it is possible to model the inner part of the boundary layer. The implemented Wall-Modeled implicit LES (WMiLES) approach replaces the no-slip boundary condition with a slip wall and a modeled shear stress. The prescribed stress results from solving a velocity profile using flow variables at a specific distance from the wall [1, 2, 3]. The model was implemented in an explicit entropy-stable dG solver that seeks the solution for the set of conservative variables while it uses the L_2 projection on the entropy variables to assemble the spatial discretization [4]. This choice aims to provide a good balance between robustness and performance. In fact, by using an orthonormal basis defined in the physical space, the mass matrix reduces to the identity when conservative variables are employed. The details of this entropy-conserving/stable framework and all the relevant references are included in [5]. Numerical results will be presented to demonstrate the conservation properties of the underlying scheme and assess the accuracy of the WMiLES solver.

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Very high-order accurate finite volume scheme for the streamfunction-vorticity formulation of incompressible fluid flows with polygonal meshes on arbitrary curved boundaries

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Keywords: Incompressible fluid flows, Streamfunction-vorticity formulation, Finite volume method, Very high-order of convergence, Arbitrary curved boundaries, Immersed boundaries

ABSTRACT

The conventional mathematical models to simulate incompressible fluid flow problems are formulated based on the Navier-Stokes equations written in terms of pressure and velocity. In that regard, the pressure-velocity coupling is a fundamental issue and, for decades, researchers have developed uncountable numerical techniques and methods to efficiently and accurately solve these equations. In two dimensions, a different approach consists in rewriting the Navier-Stokes equations in terms of two scalar quantities, the streamfunction and vorticity, related through a Laplacian operator. Compared to the conventional approach in terms of primitive variables, such a formulation does not require pressure to be computed and, therefore, the inherent difficulties associated with the pressure-velocity coupling are avoided. On the contrary, prescribing boundary conditions for the streamfunction and vorticity becomes challenging, both on the mathematical and numerical sides. The present work proposes a simple, efficient high-order accurate finite volume discretisation of the two-dimensional incompressible Navier-Stokes equations in the streamfunction-vorticity formulation. A careful discussion is devoted to deriving appropriate boundary conditions and their numerical treatment, including on arbitrary curved boundaries. For that, the conventional treatment of curved boundaries relies on generating curved meshes to eliminate the geometrical mismatch between the physical and computational boundaries and achieve a high order of convergence. However, such an approach requires sophisticated meshing algorithms, cumbersome quadrature rules on curved elements, and complex non-linear transformations. To overcome these difficulties, the reconstruction for off-site data (ROD) method [1] is employed, allowing arbitrary curved boundaries to be approximated with linear piecewise elements, while polynomial reconstructions with specific linear constraints are computed to fulfil the prescribed boundary conditions. Several benchmark test cases of incompressible fluid flow problems in non-trivial two-dimensional curved domains are addressed, and the proposed method effectively achieves up to the sixth-order of convergence.

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A first-order hyperbolic reformulation of the Cahn-Hilliard equation

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Keywords: Cahn-Hilliard equation, First order hyperbolic systems, Relaxation models, augmented Lagrangian approach, Finite Volume schemes.

ABSTRACT

We present a new first-order hyperbolic reformulation of the Cahn-Hilliard equation. The model is obtained from the combination of augmented Lagrangian techniques proposed earlier by the authors of this paper, with a classical Cattaneo-type relaxation that allows to reformulate diffusion equations as augmented first order hyperbolic systems with stiff relaxation source terms. The proposed system is proven to be hyperbolic and to admit a Lyapunov functional, in accordance with the original equations. A new numerical scheme is proposed to solve the original Cahn-Hilliard equations based on conservative semi-implicit finite differences, while the hyperbolic system was numerically solved by means of a classical second order MUSCL-Hancock-type finite volume scheme. The proposed approach is validated through a set of classical benchmarks such as spinodal decomposition, Ostwald ripening and exact stationary solutions.

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Acknowledgements

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Semi-implicit numerical scheme for hyperbolic problems

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Keywords: Finite volume method, Semi-implicit numerical scheme, Kinetic scheme, Relaxation method, Deferred Correction, non-uniform mesh.

ABSTRACT

Numerical methods for fluid-structure interaction are used in many fields of application. In explicit methods, stability is ensured by the CFL criterion, which depends on the local sound velocity. This criterion severely penalizes the time step due to the large differences in wave velocities between fluids and solids. Similarly, implicit methods that do not use the CFL condition are very costly in terms of computation time due to the cost of inversion of the system matrix.

In this presentation, we focus on the implementation of a semi-implicit numerical method for this problem. The idea is to transform a system of nonlinear conservation laws into a linear system in which the nonlinearity is reflected in the source term using a relaxation method (kinetic scheme). Time integration is performed using the Deferred Correction method [3]. Abgrall and Torlo propose such a semi-implicit, matrix-free method of any order in [1, 2] in 1D on a uniform mesh. It enables a CFL condition of a few units to be achieved without having to invert any matrix. We will present two versions of the scheme (Finite Difference and Finite Volume) of arbitrarily high order on non-uniform 1D meshes. These schemes are stable and convergent, and we will present their properties on uniform and non-uniform meshes. We will apply this method to the hyperelasticity model and build its Lagrangian version in order to extend it to fluid-structure applications. The aim is to extend it to multi-D unstructured meshes.

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On limiting for the Active Flux methods for hyperbolic conservation laws

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Keywords: hyperbolic conservation laws, finite volume method, active flux, flux vector splitting, bound-preserving, convex limiting, scaling limiter

ABSTRACT

The Active Flux (AF) method is a compact high-order finite volume method that evolves cell averages and point values at cell interfaces independently. Within the method of lines framework, the point value can be updated based on Jacobian splitting (JS), incorporating the upwind idea. However, such JS-based AF methods may encounter transonic issues for nonlinear problems due to inaccurate upwind direction estimation. This talk will present how to use flux vector splitting for the point value update, offering a natural and uniform remedy to the transonic issue.

Next we turn to limiting for the AF method. To improve robustness, our bound-preserving (BP) AF methods will be presented. Two cases are considered: preservation of maximum principle for the scalar case, and preservation of positive density and pressure for the compressible Euler equations. The update of the cell average in high-order AF methods is rewritten as a convex combination of the original high-order fluxes and robust low-order (local Lax-Friedrichs (LLF) or Rusanov) fluxes. The desired bounds are enforced by choosing the appropriate convex combination. We use a similar blending strategy for the point value update. Several challenging benchmark tests are conducted to verify the accuracy, BP properties, and shock-capturing ability of our methods, see Fig. 1.

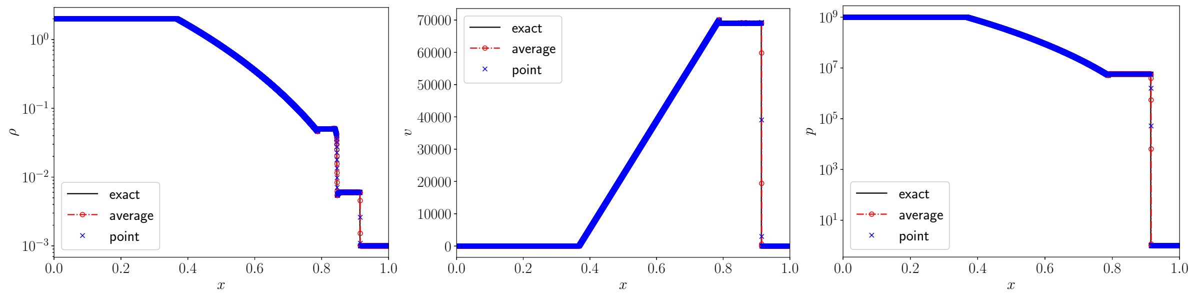

Figure 1: The density, velocity, and pressure of the LeBlanc Riemann problem computed with the bound-preserving Active Flux method on a uniform mesh of 6000 cells. The power law reconstruction and LLF flux vector splitting are used in the point value update.

Towards full Boltzmann simulations of complex fluid flows via high-order, discretely-conservative numerical schemes

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Keywords: high order numerical methods, Boltzmann equation, molecular gas dynamics, structure-preserving

ABSTRACT

The governing equations for the majority of computational fluid dynamics solvers rely on the continuum assumption for the fluid. For many problems of interest, including rarefied gases, hypersonic flows, and microflows, this assumption starts to break down, and it becomes necessary to revert to the governing equations of molecular gas dynamics which underpin the macroscopic behavior of the fluid. One such example, the Boltzmann equation, provides a statistical description of particle transport and collision which can recover the hydrodynamic equations in the asymptotic limit while providing a more detailed description of non-equilibrium systems and flows outside of the continuum regime. However, due to its high-dimensional nature, solving the Boltzmann equation comes at a computational cost that can be orders of magnitude higher than the associated transport equations for the macroscopic flow variables. As such, directly solving the Boltzmann equation has typically been limited to simpler, two-dimensional flows, and its application to more practical, three-dimensional flows has been widely considered to be intractable. To ultimately extend to these types of flows, it is necessary to both decrease the required resolution of the simulations to mitigate the rapidly increasing computational cost of the high-dimensional equation and to improve upon the computational techniques used for simulation.

In this talk, I will discuss some recent advancements in numerical methods for solving deterministic transport problems governed by the Boltzmann equation. The first is the use of high-order accurate structure-preserving discontinuous spectral element methods for unstructured grids, which allow for higher-fidelity approximations that reduce the required spatial resolution and guarantee preservation of known physical constraints (e.g., positivity of probability density, macroscopic density/pressure, etc.). This is then combined with a discretely-conservative and entropy-satisfying velocity model which allows for arbitrary velocity space discretizations while avoiding the unnecessary additional resolution typically required to ensure conservation of the macroscopic flow state. Finally, this numerical approach allows for a highly-efficient implementation on modern massively-parallel GPU computing architectures, which make it entirely possible to simulate unsteady, three-dimensional flows on modern hardware across both the rarefied and continuum regimes through direct approximation of the molecular gas dynamics equations. This method will be shown for a variety of problems ranging from wall-bounded transitional/turbulent flows to hypersonic re-entry flows.

Structure Preserving Hybrid Finite Element - Finite Volume for MHD

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Keywords: hybrid finite-volume finite-element, structure preserving schemes, semi-implicit, FEEC, compressible viscous and resistive MHD, all Mach.

ABSTRACT

In this talk we present our last recent attempts in designing a novel and efficient numerical method for the compressible viscous and resistive MHD equations for all Mach number regimes. The time-integration strategy is a semi-implicit splitting, combined with a hybrid finite-volume (FV) and finite-element (FE) discretization in space. The non-linear convection is solved by a robust explicit FV scheme, while the magneto-acoustic terms are treated implicitly in time. As a direct consequence, the resulting CFL stability condition is based only on the fluid velocity, and not on the Alfvénic and acoustic modes. The magneto-acoustic terms are discretized by compatible Finite Elements based on a continuous and a discrete deRham complexes designed using Finite Element Exterior Calculus (FEEC). Thanks to the use of FEEC, energy stability, magnetic-helicity conservation and the divergence-free conditions can be preserved also at the discrete level. A very efficient splitting approach is used to separate the acoustic and the Alfvénic modes in such a fashion that the original symmetries of the PDE governing equations are preserved even at the discrete level. In this way, the algorithm is based on the solution of linear, symmetric and positive definite algebraic systems, that are very efficiently handled by the simple matrix-free conjugate-gradient method and, at the moment, even without preconditioning. The resulting algorithm showed to be robust and accurate in low and high Mach regimes even at large Courant numbers. Non-trivial tests are solved in one-, two- and three- space dimensions to confirm the robustness, accuracy, and the low-dissipative and conserving properties of the final algorithm. While the formulation of the method is very general, numerical results for a second-order accurate FV-FEEC scheme will be presented. Higher order accurate implementation of the presented method is currently under development and, if ready, some preliminary results will be shown.

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A unified SHTC multiphase model of continuum mechanics

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Keywords: Compressible multiphase flows, Symmetric Hyperbolic and Thermodynamically Compatible (SHTC) theory

ABSTRACT

The idea of a unified approach to continuum mechanics, which is capable of describing the behaviour of different states of matter in a single system of governing equations is very attractive, especially in a multiphase context. Indeed, a continuum mixture theory imposes a big challenge if one tries to generalize it to problems involving the interaction of solids and fluids. In recent years, a unified Eulerian model of continuum mechanics that allows a unified description of all states of matter has been developed [2, 3]. In particular, the SHTC class includes a multi-fluid model developed by Romenski et al. in [4].

This talk, which is based on [1], aims to describe how to combine the two approaches and to develop a unified model of continuum mechanics for multiphase problems with an arbitrary composition of phases that may include heat conducting inviscid and viscous fluids, as well as elastoplastic solids. Furthermore, it will be shown how such a model can be numerically solved within a diffuse interface approach. Through various applications on compressible multiphase problems (Fig.1), the capabilities of this unified approach will be shown.

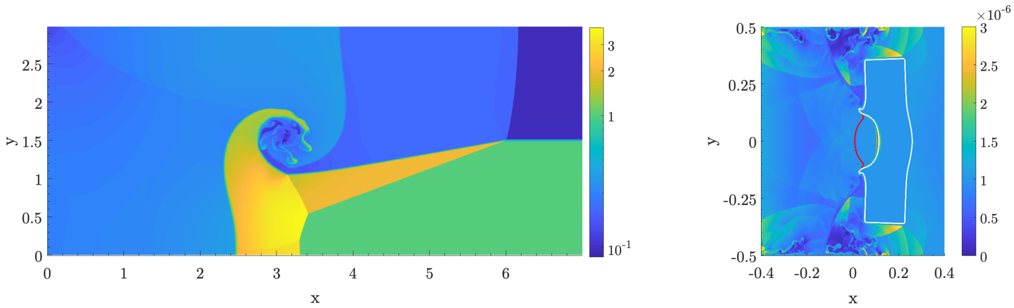

Figure 1: Multiphase and multi-material tests. The triple point problem (left), contour levels of the mixture density. A solid impact (right), the contour levels of the gas pressure.

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Compact implicit numerical schemes for nonlinear hyperbolic systems

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Keywords: high-resolution numerical methods, higher-order numerical schemes, Lax-Wendroff procedure

ABSTRACT

We present a numerical method for the solution of some hyperbolic systems including shallow water equations with nonflat topography. The method is based on implicit-explicit finite difference schemes in a combination with partial Lax-Wendroff procedure to obtain algebraic systems of equations with a convenient structure of the Jacobi matrix due to the compact stencil of the numerical scheme. The method is unconditionally stable and up to third order accurate for smooth solutions. In the case of non-smooth solutions, a high resolution form can be used based on TVD limiters with an additional time limiter for large CFL numbers. The high-resolution discretization method leads to nonlinear algebraic systems even for linear PDEs, but they can be solved using iterative fast sweeping method and deferred correction method with a very small finite number of iterations. In the case of nonlinear PDEs, a small system of nonlinear algebraic equations per each grid node must be solved in each iteration. Several numerical experiments will be presented that illustrate attractive properties of the method for some representative hyperbolic problems including well-balanced results for a small perturbation of moving equilibria in shallow water systems.

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A primitive-conservative ADER-DG method for the multimaterial Euler equations on fixed and moving polygonal meshes

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Keywords: high order numerical methods, multimaterial flows, polygonal meshes, ALE methods

ABSTRACT

We present a novel quasi-conservative arbitrary high order accurate ADER discontinuous Galerkin (DG) method allowing to efficiently use a non-conservative form of the considered PDE system, so that it can be modeled directly in the *most physically relevant* set of *variables* [1]. This is useful especially for *multi-material* flows with moving interfaces and strong contact discontinuities, as well as in presence of very non-linear thermodynamics. Regrettably, the non-conservative formulation introduces a conservation error which would normally lead to a wrong approximation of shock waves. Hence, to numerically deal with shock waves, we exploit the framework of the so-called *a posteriori* subcell finite volume (FV) limiter, so that, in shock-triggered troubled cells, we can incorporate a local conservation correction. Moreover, we provide a *local quasi-conservation condition*, which allows us to prove a *modified Lax-Wendroff theorem* valid for our approach.

To prove the capabilities of our novel approach, first, we show that we are able to recover the same results given by conservative schemes on classical benchmarks for the single-fluid Euler equations. We then conclude the presentation by showing the improved reliability of our scheme on the *multi-fluid Euler system* on examples like the interaction of a shock with a helium bubble for which we are able to avoid the development of any spurious oscillations, see the figure here below.
Some extensions to the Arbitrary-Lagrangian-Eulerian framework will be also presented.

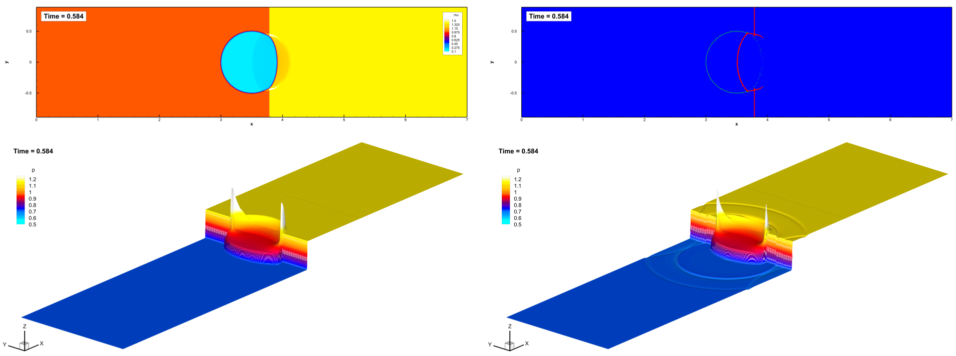

Figure 1: Shock helium-bubble interaction discretized by our DG quasi-conservative approach (on blue and green cells of top-right image) with FV conservative corrections (on red cells). Our approach avoids any spurious pressure oscillations (bottom left) unlike standard conservative methods (bottom right).

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Second and third order finite volume numerical scheme for the 2D SWE in spherical coordinates with non-constant Coriolis force. Asymptotically well-balanced for the quasi-geostrophic equilibrium

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Keywords: high order, asymptotic well-balanced schemes, finite volume methods, geostrophic equilibrium.

ABSTRACT

Trans-oceanic long waves can be modelled using the 2D Shallow Water Equations in spherical coordinates. The Coriolis force plays a major role when simulating these kind of long-lasting tsunami-like waves, as its influence may provoke a convergence into a stationary state known as geostrophic equilibrium. The geostrophic equilibrium is a steady state that occurs when the Coriolis force finds balance with the pressure gradient. It is well known that, at large scales, geophysical flows are often perturbations of this stationary solution.

Second and third order finite volume numerical schemes have been developed for the 2D SWE in spherical coordinates using MUSCL and CWENO spatial reconstructor operators, together with a TVD Runge-Kutta time integrator. The main novelty in these numerical schemes is the asymptotic preservation of the spherical version of the *quasi-geostrophic equilibrium* –a handier version of the geostrophic equilibrium maintaining its core properties. The quasi-geostrophic equilibrium can be derived linearizing the equations with a flat bottom around a background still state. At large scales, the small perturbations ϵ are of the order of the Froude and Rossby numbers ($\epsilon = 10^{-2}$ approx.). By reconstructing a local quadratic (second order scheme) or cubic (third order scheme) free surface whose gradient is in balanced with the local Coriolis force, we prove that the schemes are well-balanced as ϵ tends to zero, with order $O(\epsilon^2)$.

The code is implemented in CUDA language, making use of GPU-accelerated hardware. Results of several numerical tests demonstrating the enhanced accuracy when dealing with perturbations of some geostrophic equilibriums will be presented. The final objective of this work is to simulate the meteotsunami phenomena.

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Preserving non-moving steady states for the Euler equations of gas dynamics with gravitational forces and the Ripa model

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Keywords: Hydrostatic stationary solutions, well-balanced methods, high-order methods, reconstruction operators, Euler equations with gravity, Ripa model

ABSTRACT

Finite-volume numerical methods have been suggested in literature for solving the compressible Euler equations of gas dynamics under the influence of gravitational forces, as well as for the Ripa model. The majority of these methods aim to preserve specific hydrostatic stationary solutions or discrete approximations of all of them (see, for instance, [1] and [2]). Our goal here is to extend these two strategies in order to obtain high-order finite-volume methods that preserve approximately all the hydrostatic stationary solutions and exactly a 2-parameter family of them for a class of systems of balance laws that includes both the Euler equations and the Ripa model. In particular, numerical methods which not only preserve approximately all hydrostatic stationary solutions, but also preserve exactly some barotropic solutions of the Euler system or those corresponding to constant free surface for the Ripa model will be introduced. These schemes will be denoted by EFWB methods. In addition, the two strategies above (preserving exactly a 2-parameter family or preserving approximately all hydrostatic steady states) can be seen as particular cases of EFWB methods.

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Arbitrary high-order two-dimensional composite finite volume schemes with induced physically admissible reconstruction

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Keywords: composite numerical fluxes, high-order numerical methods, induced (admissible) reconstruction.

ABSTRACT

We deal with extension of the finite volume classical ARS (Approximate Riemann Solver) of conservation laws in dimension $d = 2$:

$$\partial_t \mathcal{U} + div_{\mathbf{x}} \mathcal{F}(\mathcal{U}) = 0, \qquad \mathbf{x} \in \mathbb{R}^2.$$

We consider composite finite volume schemes in the sense that the numerical fluxes are defined at all co-dimension object (nodes and edges here) on some quadrature points of cell boundary. For each of these mesh items, we propose a construction of local normals in such a way to fullfill (some necessary) LSV conditions: (Local conservation, Stokes formula and cell Volume formula). An high-order condition enable us to deal with arbitrary order quadrature composite formula.
For the two dimensional case (see [1]), we proposed a nodal definition of some classical edge ARS (Roe, VFFC, Rusanov, ..), then on composite edges/nodes fluxes with properties of beeing consistent and locally conservative.
For Euler equations of compressible gas dynamics (with a perfect gas law), In order to obtain a valid high order spatial reconstruction, we deal with the arbitrary order admissible induced limitation process [2]. We propose to limit some variable in order to fulfill some principles (physical or invariant domain validity) in such a way that it induce the limitation of some other variables. Here, we apply this strategy in the case of compressible Euler system of gas dynamics. In this case, we focus on the construction of the velocity limitation induced by the limitation of specific internal energy (and density).
We show some comparison on second, third and fourth order schemes to asses the accuracy and robustness of the approach.

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High-Order Positivity-Preserving Methods for Hyperbolic Balance Laws

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Keywords: High-order nonlinear numerical methods, Positivity-preserving and conservative schemes.

ABSTRACT

In the context of partial differential equations (PDEs), positivity preservation is usually obtained within a numerical method by a positivity preserving reconstruction and an additional time step constraint. In the case of the compressible Euler equations both mass and energy must remain positive, whereas the momentum is not subject to any such condition. The pressure is also determined by a known equation of state, which is used together with the total mass for the calculation of the speed of sound. If either the mass or the pressure becomes negative in the course of the numerical calculations, this results in a complex speed of sound and the method fails. One way to counteract this is by using unconditionally positivity-preserving methods such as modified Patankar (MP) schemes, including MP Runge–Kutta (MPRK) [1, 2, 3], MP strong-stability-preserving Runge–Kutta (SSPMPRK) [4, 5] and MP Deferred Correction (MPDeC) methods [6], to name a few. The resulting MP schemes are provable unconditionally positivity-preserving, yet highly nonlinear. These methods have only recently been applied in the context of the shallow water equations for guaranteeing a positive water height while being well-balanced [7], or the reactive Euler equations [4, 5]. This talk is dedicated to the introduction of MP methods in the context of PDEs including the application to the compressible Euler equations and efficiency comparisons.

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An Efficient Discontinuous Galerkin Spectral Element Implementation on Heterogeneous Grids

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Keywords: high order numerical methods, triangular elements, heterogeneous grids.

ABSTRACT

The discontinuous Galerkin spectral element method (DGSEM) has proven to be a highly efficient numerical scheme for the approximation of smooth multiscale problems, e.g., turbulence. Key aspects for the high efficiency are its excellent parallelization properties and the use of a tensor-product ansatz, where the multidimensional space operator is decomposed into a sequence of one-dimensional operations. In its classical tensor-product formulation, the DGSEM was first only developed for quadrilateral and hexahedral elements. However, for the discretization of complex geometries more flexible element types, e.g., simplexes, are in general more favorable. A common approach to benefit from the computational efficiency of the tensor-product ansatz on, e.g., tetrahedral elements is the use of a collapsed coordinate transformation, see e.g. [1, 2, 3]. An intrinsic disadvantage of the collapsed coordinate transformation is a smaller time step in comparison to a full-order basis. This is induced by the clustering of degrees of freedom in the vicinity of the collapsed faces. This shortcoming was recently resolved in [3] by using modal degrees of freedom instead of nodal ones, where a weight-adjusted approximation of the inverse of the modal mass matrix was utilized. However, in practical applications the use of hexahedral elements in the larger part of the computational domain is favorable and only in a small subset simplexes are required.

Thus, in the present work, we present the extension of the approach discussed in [3] to triangular prism elements and a heterogeneous discretization composed out of triangular and quadrilateral or triangular prism and hexahedral elements. Here, the degrees of freedom of the simplex elements are evolved in the modal and the ones of the quadrilateral and hexahedral elements in the nodal space. We prove the free-stream preservation and the convergence properties based on numerical experiments and show that the excellent scaling properties of the DGSEM are preserved. To demonstrate the capabilities of the approach, a three-dimensional large eddy simulation of a transonic airfoil discretized with heterogeneous elements is performed. Finally, the extension of the approach to fully unstructured three-dimensional problems is discussed.

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On a semi-discrete Active Flux method for multi-dimensional conservation laws

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Keywords: Active Flux method, high order finite volume scheme, structure preserving method, multi-dimensional schemes.

ABSTRACT

The quest for numerical methods for conservation laws that take into account multi-dimensional flow structures motivate us to pursue the Active Flux method. A semi-discrete AF method for two space dimensions is described in [1]. It is a third order finite volume method with additional degrees of freedom, namely point updates at points that are shared between neighboring cells. Thus the reconstructed solution is continuous between cells.

We begin this lecture by introducing this semi-discrete Active Flux method. We shall discuss how Active Flux will resolve some multi-dimensional flow features well, even on a coarse grid. To this end we begin with the study of linear acoustics. On a cartesian grid in two space dimensions using Fourier analysis we can identify the class of those schemes that preserve discrete stationary states, in particular well-balanced schemes. For the acoustic equations the notion of a low Mach limit exists and it can be seen that the stationarity preserving schemes are automatically low Mach. This way one can see that the Active Flux method has these structure preserving properties.

For the non-linear Euler equations the ability of Active Flux to maintain stationarity solutions and to resolve vortices well, also on a coarse grids, is shown by numerical experiments. The method also has the low Mach property for the Euler equations. Note that there are no fixes or modifications of Active Flux needed to achieve these properties. Taking the incompressible limit of the compressible Euler equations one arrives at a divergence free velocity field, which is non-trivial only in more than one space dimension. The Active Flux method has the ability to maintain both a solenoidal and a irrotational velocity field, which are multi-dimensional flow features.

We end by giving an outlook on high order extensions of the method, and on limiting, as presented by Lisa Lechner and Junming Duan in the lectures following this one.

The above list of collaborators is partial.

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Troubled-cell detection for high-order methods on 2D unstructured meshes by Convolution Neural Networks

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Keywords: Discontinous Galerkin, troubled cell indicator, unstructured meshes, neural networks

ABSTRACT

We present advancements in high-order numerical methods for approximating solutions of non-linear conservation laws, focusing on the development of an early detection mechanism where the numerical solution loses regularity, e.g., develops non-physical numerical oscillations near discontinuities. The methodology comprises two key components: (a) a discontinuity detector/troubled cell indicator for identifying problematic computational cells, and (b) an a-posteriori Finite-Volume limiter for mitigating numerical oscillations following [1, 2]. Along this line, we have adopted the ADER Discontinuous Galerkin over unstructured triangular meshes as our core numerical model [1, 2]. The key difference in our approach is that we utilize techniques from the area of Machine Learning, using a trained neural network as a troubled cell indicator, instead of numerical admissibility detection criteria which require the specification of problem-dependent parameters.

Motivated by the works in [3, 4], we have trained a Fully Connected Multi-layer Perceptron (MLP) and a Convolutional Neural Network (CNN) as binary classifiers. What differentiates our work from that in [4] is that the CNN architecture, traditionally requiring rectangular inputs, is innovatively adapted for use on unstructured triangular meshes. This adaptation involves a preprocessing step, converting triangular mesh data into a CNN-compatible format. Further more, since CNNs are inherently shift-invariant but not rotationally invariant, we enhanced the model's robustness by augmenting the training dataset. We incorporated rotations and reflections symmetrically, enabling the binary classifier to achieve rotational invariance and thus improve its discontinuity detection capabilities.

We will share preliminary results in two spatial dimensions. To our knowledge, this is a pioneering effort to apply CNNs on unstructured triangular meshes for discontinuity detection within numerical solutions. By adopting data augmentation techniques, involving rotations and reflections of the training samples, we enabled the traditional CNN architecture to possess the necessary symmetry to function effectively as a binary classifier in this novel context.

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A Curvilinear Euler–Lagrange Code on Unstructured Moving Meshes

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Keywords: high order numerical methods, Euler-Lagrange, moving mesh methods.

ABSTRACT

Flows with dispersed solid or liquid particles suspended in the ambient medium are ubiquitous both in nature and in engineering applications. In the latter case, the dispersed multiphase flow is frequently subjected to time-varying geometric boundary conditions to represent the technical work performed on or extracted from the fluid. Furthermore, these geometries are generally complex, while the occurring high Reynolds numbers result in turbulent fluid fields which comprise a wide range of active flow scales.

High-order methods based on an Euler–Lagrange representation of the different phases pose an enticing approach for the numerical simulation of such problems due to their inherently low dissipation and dispersion errors. However, preserving the conservation properties of the Eulerian phase together with water-tight tracking on the Lagrangian phase on curved, unstructured and moving domains is generally challenging. Here, we present recent extensions of the open-source, high-order accurate CFD framework ∃LEXI [1] which represent our endeavor to combine a Discontinuous Galerkin Spectral Element Method (DGSEM) with particle tracking capabilities on unstructured hexahedral time-varying grids. Through formulation of the conservation equations in arbitrary Lagrangian-Eulerian (ALE) formulation [2], ∃LEXI can incorporate both rigid body movement and mesh deformation. The combination with an efficient, high-order accurate sliding mesh [3] approach enables the conservative coupling of multiple subdomains. Special care is taken to retain the favorable scaling properties of the baseline framework throughout the implementation. This work concludes by presenting examples of large scale computations for dispersed multiphase flows in complex systems and giving an outlook on the next research challenges.

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Limiters for the Discontinuous Galerkin Method on Quadrilateral Meshes

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Keywords: high order, limiting, quadrilateral meshes, discontinuous Galerkin methods.

ABSTRACT

Limiting techniques are used to stabilize numerical solutions of hyperbolic conservation laws in the presence of discontinuities. Most limiters are developed for Cartesian or triangular meshes and relatively less attention has been paid to quadrilateral meshes. Such meshes can be of practical interest as they have a smaller number of degrees of freedom for a given domain. At the same time they are more challenging to perform analysis on due to nonlinear mapping, lack of orthogonality of the computational basis, etc.

We present a family of limiters for the discontinuous Galerkin method on unstructured quadrilateral meshes. The technique involves decoupling solution coefficients as directional derivatives along two chosen directions and limiting them along said directions using a one-dimensional minmod slope limiter. The directional derivatives are reconstructed using solution in a compact neighborhood stencil of eight elements which can be computed in the preprocessing stage. Further, we provide a range of scaling parameters for which the limited solution satisfies the local maximum principle in means. We also derive a new, tight CFL condition that allows to increase the time step size. Finally, we present numerical examples to confirm that the limited solutions retain the theoretical rate of convergence and resolve discontinuities.

A Well-Balanced Fifth-Order A-WENO Scheme Based on Flux Globalization

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Keywords: A-WENO schemes, flux globalization based path-conservative central-upwind scheme, nozzle flow system, two-layer shallow water equations.

ABSTRACT

We construct a new fifth-order flux globalization based well-balanced (WB) alternative weighted essentially non-oscillatory (A-WENO) scheme for general nonconservative systems. The proposed scheme is a higher-order extension of the WB path-conservative central-upwind (PCCU) scheme recently proposed in [1]. We apply the new scheme to the nozzle flow system and the two-layer shallow water equations. We conduct a series of numerical experiments, which clearly demonstrate the advantages of using the fifth-order extension of the flux globalization based WB PCCU scheme.

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High-resolution compact semi-implicit level set methods for the advection equation

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Keywords: level set methods, compact schemes, advection equation, semi-implicit

ABSTRACT

This study focuses on the development of innovative semi-implicit finite-difference schemes tailored for solving the advection equation on Cartesian grids. By employing both the Lax-Wendroff [1] and the inverse Lax-Wendroff methods [2], compact numerical schemes are derived, leading to unconditional stability and high accuracy. The resulting stencil can span multiple time levels, incorporating even future points in time [3]. Additionally, the compact nature of the stencils allows for the utilization of efficient solvers, such as the fast sweeping method, resulting in a highly efficient semi-implicit approach where the new solution is explicitly derived from previously computed values. The schemes presented achieve up to third-order accuracy for smooth solutions while demonstrating no non-physical oscillations in the solution gradient for the high-resolution form.

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A two-dimensional Active Flux method of arbitrarily high order

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Keywords: multi-dimensional conservation laws, high order methods, semi-discrete Active Flux.

ABSTRACT

The Active Flux method is an extension of a finite volume method using point values at the cell interfaces in addition to cell averages as degrees of freedom. Since these point values are shared between adjacent cells, a globally continuous solution is obtained. The method was introduced as a fully discrete scheme of third order, see e.g. [4]. Here, instead of relying on Riemann solvers, evolution operators are used to actively update the point values at the cell interfaces making the method multi-dimensional by conception. More recently, a semi-discrete version of the Active Flux method was presented in [2, 3], which is directly applicable to different systems of conservation laws. In *one* spatial dimension this scheme was extended to arbitrary order of accuracy (see [1, 2]). In particular, a compact scheme using *moments* was introduced, yielding a hybrid finite element–finite volume method.

In this talk we present such a *generalized Active Flux* method of arbitrarily high order of accuracy in *two* spatial dimensions on Cartesian grids. It uses a conservative update of the cell averages and its higher moments, whereas the point values, for which exists no notion of a conservative update, are updated using a non-conservative form of the hyperbolic PDE. Numerical experiments with the extended scheme will be shown to illustrate the capabilities of the method.

This work is done among others with W. Barsukow and C. Klingenberg.

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R-Block structural schemes for ordinary differential equations

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Keywords: high order numerical methods.

ABSTRACT

We present a compact scheme whose core concept involves decomposing it into two subsystems of equations. The Physical Equations utilise the function and its K derivatives at a node by implementing physical relations. These equations operate locally, with no exchange of information with other nodes, as the physics involved are governed by local operators. The Structural Equations depend on linear relationships between the function and its derivatives across a stencil of R points, which we call a R-block, establishing complete connections between a node and its neighbours. These relationships are independent of the physics involved since they are established regardless of the specific problem.

In this presentation we address in particular the accuracy and stability of these methods.

Hybrid High-Order methods with hybrid pressure and improved turbulence modelling capabilities

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Keywords: Hybrid High-Order, Navier–Stokes equations, Pressure-robust, Pointwise divergence free.

ABSTRACT

In recent years, the introduction of Hybrid High-Order (HHO) and Hybridizable Discontinuous Galerkin (HDG) methods has provided a new ground to pursue the development of high-order accurate computational modelling tools for the simulation of incompressible turbulent flows. In contrast to Discontinuous Galerkin (DG) methods, HHO and HDG methods are based on degrees of freedom (DoF) that are broken polynomials on both the mesh and its skeleton. Relevant features of hybrid schemes are: i) local (element-by-element) conservation of physical quantities, ii) increased convergence rates, iii) robustness with mesh distortion and grading, iv) implicit time integration with reduced memory footprint of Jacobian matrix compared to DG. In addition, by introducing physics dependent local reconstructions of discrete operators, hybrid formulations enable the possibility to reproduce key continuous properties at discrete level.

The key property that the irrotational part of body forces only affects the pressure, leaving the velocity field unaltered, has been achieved by introducing hybrid pressure spaces that are able to provide pointwise divergence free velocity fields, continuity of the normal traces of the velocity over mesh faces and inf-sup stability, without requiring pressure stabilization [1]. Alongside, non-dissipative convective term formulations can be devised, ensuring that the discretization of the convective term does not alter the balance of kinetic energy in case of passive boundary conditions [2]. A HHO scheme featuring both the aforementioned properties, namely pressure-robustness and kinetic-energy-preservation, has been proposed in [3].

Interestingly, we recently found out that numerical dissipation can be introduced in the new scheme by upwinding the tangential component of the velocity at interelement boundaries, allowing to cope with the inviscid limit. Even if kinetic energy preservation is lost, the improved robustness is essential for tackling turbulence modelling at high Re numbers. Comparison with state-of-the-art DG formulations based on Godunov fluxes on inviscid and viscous test cases has shown that the new scheme features a lower numerical dissipation. In this work we propose to further improve the HHO scheme with hybrid pressure introducing high-order ESDIRK temporal schemes and comparing different convective trilinear term formulations. Moreover we plan to further investigate its capabilities on challenging test cases, possibly tackling under-resolved turbulence modelling.

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Pushing the Geometrical Capabilities of High Order Galerkin Spectral Element Methods

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Keywords: hp-adaptive, discontinuous Galerkin, spectral element, immersed boundary.

ABSTRACT

High order spectral element methods have gained efficiency and popularity in the last thirty years as computational resources have grown and sustained research efforts have been devoted to their development. To date, however, there is no tailored grid generator for these methods and many of the geometries of studies using these methods remain canonical. This presentation will explore several methodologies to allow for more complex geometries in simulations using high order methods.

Once a complex geometry is captured, either by spline representation or by defined analytical expressions, hp-adaptivity can be used to develop adequate starting grids for calculations and to judiciously adjust grid resolution to precisely track solutions as they develop in time. Adaptivity is guided by *a posteriori* error estimators that use the spectrum of the element-local solution to estimate the error and evaluate the quality of the resolution, thereby serving as a criterion to decide between h- (element splitting/coarsening) or p- (polynomial increase/decrease) adaptivity if necessary. For large scale calculations, the dynamic adaptive procedure needs to include load balancing among processors for the calculation implementation to remain efficient. We present a parallel hp-adaptive discontinuous Galerkin spectral element solver for acoustic problems in curvilinear geometries with dynamic load balancing based on a Hilbert space-filling curve algorithm. The curvilinear geometries are treated either by high-order mappings of curved elements to the reference square or, more simply, as immersed boundaries in Cartesian grids. We introduce immersed boundaries in high order spectral element methods through volume penalization and use adaptivity (most often h- as dictated by the error estimator) to capture the solution in the vicinity of the immersed boundary. Comparisons of results from the two methods will be discussed (example shown in Fig. 1).

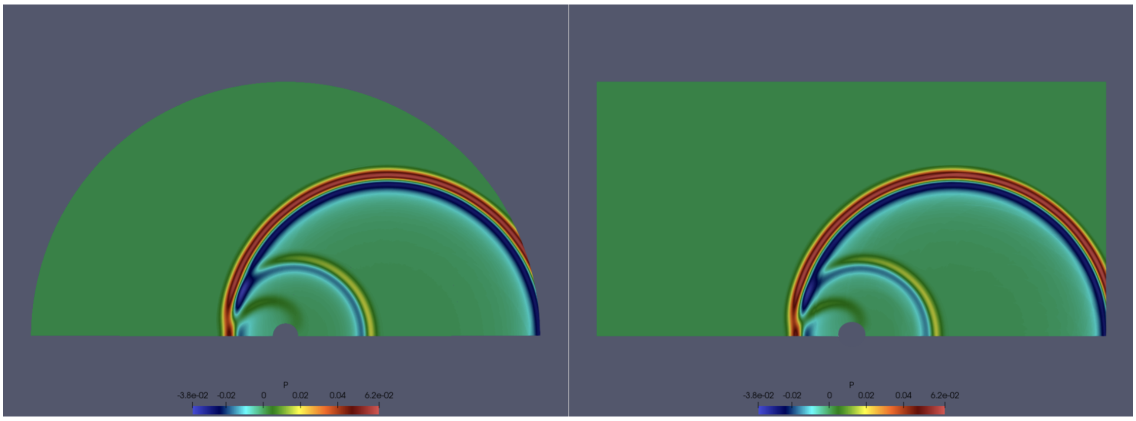

Figure 1: h-p adaptive solutions of an acoustic scattering problem of a pressure disturbance off a circular cylinder. Left: using curvilinear mappings. Right: using immersed boundaries.

Scalable Interpolation at Arbitrary Points in High-Order Volume and Surface Meshes on GPUs

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Keywords: interpolation, high-order finite elements, GPU.

ABSTRACT

Robust and scalable arbitrary point interpolation is required in the finite element method (FEM) and spectral element method (SEM) for querying the PDE solution at points of interest in the domain, comparison/remap of solution between different meshes, and Lagrangian particle tracking. In FEM and SEM, high-order funtions are typically represented in physical space on each mesh element through a map from a corresponding canonical element in the reference space. Interpolation at a given location thus relies on the element that overlaps the point and the corresponding reference space coordinates. This becomes especially challenging for large scale meshes with high performance computing (HPC).

We present *findpts*, a library for interpolation of functions in H^1 finite element space at arbitrary points on area meshes with quadrilaterals in 2D and volume meshes with hexahedra in 3D [1]. *findpts* provides robust and scalable interpolation through a sequence of three routines. First, in the `setup` step, we construct element-wise bounding boxes and a *local* and a *global* hash-table on each MPI rank. Next, in the `search` step, the element-wise bounding boxes and *local* hash-table are used to efficiently find candidate elements overlapping a given point. The Newton's method is then used to solve a minimization problem on these elements and determine the corresponding reference space coordinates. Points that are not found within any element are sent to other candidate MPI ranks determined by the *global* hash-table, where the rank-local search process is repeated and results are communicated back. Finally, any discrete high-order function is efficiently `interpolated` by leveraging the tensor-product structure of the basis functions. We have recently also extended *findpts* for surface meshes (lines in 2D and quads in 3D). This capability is crucial for applications based on manifolds and closest point search when points are known to be outside the domain (e.g., detecting contact between two bodies).

To leverage recent advances in HPC, we effect communication bounded steps such as hash-table calculation on CPUs using the native *findpts* library, and compute bounded steps such as the Newton search and interpolation on GPUs using specialized kernels. We demonstrate the effectiveness of our approach for high-order interpolation using MFEM [2, 3], an open-source high-order finite element library.

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An h-p Adaptive Strategy for Discontinuous Galerkin Schemes

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Keywords: high order, adaptivity, h-p refinement, discontinuous Galerkin scheme

ABSTRACT

The discontinuous Galerkin (DG) scheme offers a highly adaptable approach to approximating differential equations, leveraging several key properties. It can be viewed as a standard numerical approximation technique, employing polynomials within each grid cell to approximate the solution of the differential equation. Particularly for smooth solutions, utilizing polynomials of high degree yields an efficient computational procedure. Moreover, owing to its handling of the grid cell coupling via fluxes at grid cell interfaces, the scheme doubles as a domain decomposition strategy. This concept entails dividing the computational domain into subregions, the grid cells, that are weakly coupled.

These facets of the discontinuous Galerkin scheme have found applications in numerous shock-capturing methods, where the approximations in troubled grid cells are computed using a stable finite volume (FV) scheme on a sub-grid. Within this framework, we focus in this talk on the control of the switching and the FV-DG coupling. Specifically, the scheme employs a proficient p-adaptive discontinuous Galerkin operator in regions characterized by smooth flow, while shocks and interfaces are effectively captured by a finite volume scheme on an h-refined, element-local sub-grid. Consequently, the resulting hp-adaptive scheme seamlessly integrates the high-order accuracy of the DG method with the robustness of the FV scheme, employing p-adaptation in smooth regions and h-refinement at discontinuities, respectively. Precise control of the switching mechanism is paramount to prevent spurious oscillations between differently resolved regions.

Both p-refinement and the capturing of shocks and interfaces are dynamically performed at runtime, guided by an indicator rooted in the modal decay of solution polynomials. However, in parallel simulations, the hp-adaptive discretization may introduce significant imbalances in processor workloads. To address this, we propose a dynamic load balancing scheme that redistributes the workload based on element-local wall time measurements along a space-filling curve. We present numerical results illustrating the efficiency of this approach. In combination with a level-set treatment of interfaces, we indicate the scheme's capability to finely resolve also multiphase flow phenomena, including pronounced interface deformations and merging interface contours ([1]).

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An Energy Stable Nonlinear Incompressible Multi-Phase Flow Formulation

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Keywords: Multi-phase flow, volume of fluid formulation, weak boundary conditions, energy stability

ABSTRACT

We have in a series of papers [1, 2, 3], shown that a specific skew-symmetric reformulation of the shallow water equations, the incompressible Euler and Navier-Stokes equations and the compressible Euler equations can be derived. This skew-symmetric formulation was augmented with nonlinear boundary conditions such that a continuous energy estimate in terms of only external data was obtained. Following [4], we combined the skew-symmetric formulation with summation-by-parts (SBP) difference operators and weak boundary conditions (SAT) to arrive at nonlinearly stable schemes.

Here we turn our attention to IBVPs for incompressible multi-phase liquid-gas flows in the volume of fluid (VOF) setting. The VOF formulation [5] is widely used to model incompressible immiscible liquid-gas flows. It is applicable to complex interface motions and mass conservative. The method involves tracking the liquid-gas interface implicitly by advecting the volume fraction α of the target phase. If combined with a single liquid-gas velocity in mixed cells, it is referred to as a homogeneous or "one-fluid" formulation [6, 7]. We will consider an incompressible multi-phase formulation which is mathematically equivalent to a one-fluid formulation, and aim for provable nonlinear energy stability. Numerical experiments supporting the developed theory will be provided.

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Arbitrary-order finite volume schemes preserving positivity for diffusion problems on deformed meshes

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Keywords: high order, finite volume methods, positivity preserving schemes, anisotropic diffusion

ABSTRACT

In this presentation, the problem is the following

$$\begin{cases} -\boldsymbol{\nabla}\cdot(\boldsymbol{\kappa}\boldsymbol{\nabla}\bar{u}) + \lambda\bar{u} = f & \text{in} \quad \Omega, \\ \bar{u} = g_D & \text{on} \quad \Gamma_D, \\ \boldsymbol{\kappa}\boldsymbol{\nabla}\bar{u}\cdot\mathbf{n} = g_N & \text{on} \quad \Gamma_N, \end{cases} \tag{1}$$

where Ω is a bounded open domain of $\mathbb{R}^2$ with $\partial\Omega = \Gamma_D \cup \Gamma_N$ ($\Gamma_D \cap \Gamma_N = \emptyset$), and $\mathbf{n} \in \mathbb{R}^2$ is the outgoing unit normal vector. The data are such that $f \in L^2(\Omega)$, $g_D \in H^{1/2}(\Gamma_D)$, $g_N \in L^2(\Gamma_N)$, $\lambda \in \mathbb{R}^+$ (if $\lambda = 0$, then $|\Gamma_D| > 0$), and $\boldsymbol{\kappa}$ is a bounded tensor-valued coefficient that satisfies the uniform ellipticity condition:

$$\forall \mathbf{x} \in \Omega,\ \forall \xi \in \mathbb{R}^2,\ \kappa_{\min}||\xi||^2 \leq \xi^t \boldsymbol{\kappa}(\mathbf{x})\xi.$$

where $\kappa_{\min}$ is a strictly positive coefficient. Under the above conditions, one can prove that system (1) has a unique solution in $H^1(\Omega)$ that satisfies a positiveness principle, i.e. if $f \geq 0$ and $g \geq 0$, then $\bar{u} \geq 0$. One often refers to positivity preserving in the literature for this principle.

Positivity is fundamental in our applications because $\bar{u}$ can be a temperature or a concentration. We have proposed a 1D scheme [1] that preserves positivity while being arbitrary order accurate in space. As far as we know, this is the first scheme that satisfies both properties. We have shown that these schemes are conservative and positive, at the cost of the linearity of the scheme. Improving the order is achieved through polynomial reconstruction, and monotonicity is obtained by reducing to a M-matrix structure, which gives nonlinear schemes. The non-linearity is solved thanks to a fixed point algorithm. Under a stability assumption, we have also shown convergence to an order corresponding to the degree of the reconstruction. These schemes can also be symmetrized, which further induces the LMP [2] (Linear Maximum Preserving) structure of the scheme. These schemes are also adapted in the case of a discontinuous $\boldsymbol{\kappa}$ coefficient.

This talk explains how to extend our 1D monotone scheme of arbitrary order to dimension 2. We therefore propose a family of finite volume schemes of arbitrary order that preserves positivity on any mesh with a diffusion coefficient $\boldsymbol{\kappa}$ that can be discontinuous and/or anisotropic. This strain of schemes is implemented in C++ in an open platform of the CEA in order to validate this approach.

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How to preserve a divergence or a curl constraint in a hyperbolic system with the discontinuous Galerkin method

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Keywords: discontinuous Galerkin methods, hyperbolic systems, curl and divergence preservation.

ABSTRACT

When dealing with systems of conservation laws, implicit differential constraints may be encountered on vector equations. For example, if a system includes the equation

$$\partial_t \mathbf{u} + \nabla p = 0,$$

where $\mathbf{u}$ is an unknown of the system and p depends linearly or nonlinearly on the other variables of the system, then taking the curl of the equation leads formally to

$$\partial_t (\nabla \times \mathbf{u}) = 0.$$

In the same manner, if $\mathbf{u}$ follows the equation

$$\partial_t \mathbf{u} + \nabla \times \mathbf{q} = 0,$$

then the following equation holds formally

$$\partial_t (\nabla \cdot \mathbf{u}) = 0.$$

Such constraint can be encountered for example in the context of the wave system (conservation of the vorticity), or Maxwell system (conservation of zero divergence of the magnetic field), and is known to be challenging to preserve.

A usual strategy relies on the use of so-called *staggered schemes*, which themselves rely on the following two dimensional conforming discrete de-Rham complex

$$\begin{cases} \mathbb{P}_{k+1} \xrightarrow{\nabla_{\mathbf{x}}^{\perp}} \mathbb{RT}_{k+1} \xrightarrow{\nabla_{\mathbf{x}}\cdot} \mathrm{d}\mathbb{P}_k \\ \mathbb{P}_{k+1} \xrightarrow{\nabla_{\mathbf{x}}} \mathbb{N}_{k+1} \xrightarrow{\nabla_{\mathbf{x}}^{\perp}\cdot} \mathrm{d}\mathbb{P}_k \end{cases} \tag{1}$$

where $\mathbb{P}_{k+1}$, $\mathbb{RT}_{k+1}$, $\mathbb{N}_{k+1}$ and $\mathrm{d}\mathbb{P}_k$ are respectively the classical continuous finite element space, the Raviart-Thomas finite element space, the two-dimensional Nédélec finite element space and the discontinuous finite element space.

In this talk, we will show that an equivalent of (1), but involving discontinuous approximation spaces instead of $\mathbb{RT}$ and $\mathbb{N}$ can be derived, at the price of slightly increasing the number of degrees of freedom, and ensures a crucial property: the harmonic gap property. Based on these approximation spaces, we will show that divergence and curl constraint are naturally preserved under mild assumptions on the numerical flux.

Two dimensional high order numerical results involving the wave system, the Maxwell system and the induction equation will be shown, and will prove that the method is high order and preserves exactly the differential constraints.

Half-Closed Discontinuous Galerkin Discretisations

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Keywords: Discontinuous Galerkin, Sparsity, Linear solvers, Static condensation

ABSTRACT

We introduce the concept of half-closed nodes for nodal Discontinuous Galerkin (DG) discretisations. This is in contrast to more commonly used closed nodes in DG where in each element nodes are placed on every boundary. Half-closed nodes relax this constraint by only requiring nodes on a subset of the boundaries in each element, with this extra freedom in node placement allowing for increased efficiency in the assembly of DG operators. To determine which element boundaries half-closed nodes are placed on we outline a simple procedure based on switch functions. We examine the effect on operator sparsity from using the different types of nodes and show that in particular for the Laplace operator for there to be no difference in the sparsity from using half-closed or closed nodes. We also discuss in this work some linear solver techniques commonly used for Finite Element or Discontinuous Galerkin methods such as static condensation and block-based methods, and how they can be applied to half-closed DG discretisations. Finally we demonstrate its use on a range of test problems including in CFD, and benchmark its performance on these numerical examples.

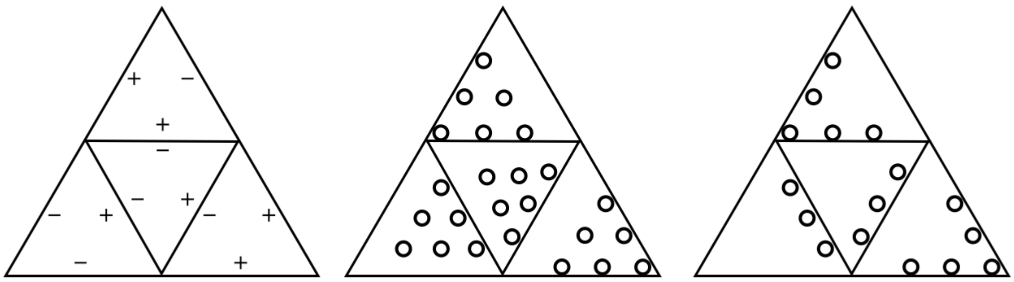

Figure 1: Static condensation elimination pattern for half-closed nodes on triangles. The left figure shows an example switch function, the middle shows shows the original half-closed nodeset, and in the right figure nodes not on boundaries where the switch $S_n^m > 0$ are eliminated with static condensation.

In-cell Discontinuous Reconstruction path-conservative methods for nonconservative hyperbolic systems - 1D and 2D extension

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Keywords: path-conservative methods, nonconservative products, high-order numerical methods, in-cell reconstructions.

ABSTRACT

We consider nonconservative hyperbolic systems of the form

$$\partial_t \mathbf{u} + \mathcal{A}(\mathbf{u})\partial_x \mathbf{u} = 0, \tag{1}$$

in which the unknown $\mathbf{u}(x, t)$ takes values in an open convex set Ω, and $\mathcal{A}(\mathbf{u})$ is a smooth locally bounded map from Ω to $\mathcal{M}_{N\times N}$. We are interested in the numerical approximation of discontinuous solutions for system (1). Due to the lack of control of numerical viscosity, standard path-conservative methods converge, in general, to weak solutions with wrong jump conditions in the presence of non-conservative products. In this talk we focus on the strategy developed in [1] and [2] based on in-cell discontinuous reconstructions which deal with this challenging topic. We explain how extend this strategy to high-order accuracy and to two-dimensional problems. This strategy captures exactly isolated shocks and seems to converge to the right solution when more than one shock appears in the solution. Some numerical results will be shown.

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Entropy-stable discontinuous Galerkin solution of the multicomponent compressible Euler model with entropy balance enforcement

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Keywords: discontinuous Galerkin, multicomponent Euler model, entropy projection

ABSTRACT

This talk presents the development of an efficient discontinuous Galerkin (dG) solver for the multicomponent compressible Euler equations. The method provides global entropy conservation/stability at the discrete level, contributing to the computations robustness, cf. [1, 2]. The unsteady term of the governing equations is formulated for the conservative variables, while the spatial discretization is assembled from their L_2 projection on the entropy ones [1, 2], as suggested by Chan et al. [3] and Alberti et al [4]. This approach requires numerical over-integration to ensure entropy conservation/stability, significantly degrading the computational performance. The Direct Enforcement of Entropy Balance (DEEB) proposed by Abgrall in [5] is implemented to avoid this. DEEB consists of an explicit correction to the discretization for unphysical entropy variations. As high-order discretizations give rise to spurious oscillations at flow discontinuities, a directional shock-capturing term is also added to the discretized equations. Numerical results for several multicomponent Riemann problems and the shock-bubble interaction flow case are used to compare the performance of our strategy to alternative approaches directly solving for conservative and entropy variables.

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Positivity preserving time implicit DGSEM for hyperbolic conservation laws

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Keywords: high order numerical methods, discontinuous Galerkin, implicit methods, positivity preserving, entropy stability

ABSTRACT

In this talk, we will review recent work on the discontinuous Galerkin spectral element method (DGSEM) with implicit time stepping for the numerical approximation of linear and nonlinear hyperbolic conservation laws in multiple space dimensions. We consider either the DGSEM with a backward Euler time stepping, or a space-time DGSEM discretization to remove the restriction on the time step. We design artificial viscosities and limiters to make the schemes positivity preserving and entropy stable. Numerical experiments will be presented to illustrate the properties of these schemes.

Numerics for compressible liquid-vapour flow: sharp-interface and diffuse-interface models

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Keywords: multi-phase flows, discontinuous Galerkin, sharp-interface, diffuse-interface, moving mesh

ABSTRACT

The mathematical modelling of compressible liquid-vapour flow on the detailed scale with separated liquid and vapour phases leads to either sharp-interface models or to diffuse-interface models. For the sharp-interface modelling in the first part we consider a free boundary value problem with Euler or Navier-Stokes equations in the bulk regions, and a special handling of the interfacial dynamics which depends sensitively on smaller-scale effects. Based on a recently introduced moving-mesh concepts for finite-volume methods we present a heterogeneous multiscale ansatz which realizes mass and energy transfer across the interface by a molecular-dynamics ansatz as a machine-learned surrogate model. Numerical simulations cover pure liquid-vapour flow and multi-component systems.

In the second part of the lecture we focus on phase field models as representative models for the diffuse-interface ansatz. We consider a compressible Navier-Stokes-Allen-Cahn system and propose a higher-order numerical scheme that fulfills the energy stability on the discrete level without introducing artificial numerical dissipation. The approach employs a discontinuous-Galerkin (dG) discretization that relies on a mixed reformulation of the governing equations. Besides discrete thermodynamical consistency the fully-discrete scheme ensures mass conservation and –most notably– second-order accuracy in space and time. Numerical experiments for typical two-phase flow regimes confirm the theoretical findings and show the applicability of the approach.

A Robust Entropy-Stable Discontinuous Galerkin Scheme for the Multi-Ion MHD System

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Keywords: high-order discontinuous Galerkin methods, entropy stability, structure-preserving methods, plasma physics

ABSTRACT

We present an entropy-stable discontinuous Galerkin (DG) scheme for the multi-ion magnetohydrodynamics (MHD) equations and efficient subcell limiting strategies to improve its robustness for challenging plasma simulations.

We start by performing a continuous entropy analysis of the multi-ion MHD system described by, e.g., Toth et al. [1], which describes the motion of multi-ion plasmas with independent momentum and energy equations for each ion species. Following the continuous entropy analysis, we propose a modification to the multi-ion MHD system, such that entropy consistency is guaranteed at the continuous level. Moreover, we augment the system of equations with a generalized Lagrange multiplier (GLM) technique to enforce the divergence-free condition on the magnetic field.

We derive robust entropy-conservative and entropy-stable finite volume (FV) fluxes for the modified multi-ion GLM-MHD system, and extend them to a high-order DG framework using collocated Legendre-Gauss-Lobatto SBP operators. These FV and DG schemes are consistent with the EC and ES schemes for the single-fluid GLM-MHD equations of Derigs et al. [2]. The resulting schemes guarantee the fulfillment of the second law of thermodynamics at the semi-discrete level, while maintaining local node-wise conservation properties. Finally, we extend the family of subcell limiting strategies for high-order discontinuous Galerkin spectral element methods (DGSEM) presented in [3] to the multi-ion GLM-MHD system.

We numerically validate the high-order convergence and entropic properties of our scheme and use it for challenging multi-species MHD simulations.

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Entropy stable shock capturing for high-order discontinuous Galerkin schemes on moving meshes

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Keywords: high order numerical methods, shock capturing, moving mesh methods.

ABSTRACT

The discontinuous Galerkin method allows a straightforward construction of efficient high-order methods on arbitrary grids for hyperbolic partial differential equations (PDEs) such as the Euler equations. However, hyperbolic PDEs admit discontinuities in the solution, even if the initial solution is smooth. Moreover, it is well known that high-order schemes induce oscillations at discontinuities, called Gibbs phenomena. Hence, adequate numerical treatment is necessary to detect and handle discontinuities, ranging from shock fitting to shock capturing approaches.

Recently, an entropy stable shock capturing method based on a convex blending of a low-order finite volume and a high-order discontinuous Galerkin method has been proposed [1, 2]. This approach exploits the fact that a high-order summation-by-parts operator can also be written in the form of a conservative finite volume scheme.

In this talk, an extension of this entropy stable shock capturing scheme to moving meshes is proposed. The mesh movement is thereby based on the arbitrary Lagrangian–Eulerian method. The freestream preservation and high-order convergence properties of the resulting scheme are demonstrated. Finally, the capabilities of the convex blending operator are demonstrated using a three-dimensional large-eddy simulation of a plunging airfoil under transonic flow conditions.

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QUINPI: going implicit for nonlinear hyperbolic equations

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Keywords: high order numerical methods, structure preserving schemes, Lagrangian methods.

ABSTRACT

Many interesting applications of hyperbolic systems of equations are stiff, in the sense that restrictive CFL conditions are imposed by fields that one is not really interested in tracking accurately. A typical solution in these cases is to resort to implicit time integration, but in the field of high order accurate numerical schemes for hyperbolic equations this is made very difficult by the extreme nonlinearity of the reconstruction operators.

In this talk I will illustrate an approach, that we called Quinpi, to treat nonlinear hyperbolic equations with high order accurate implicit timestepping. It is based on a third order Central WENO reconstruction, a third order DIRK time integrator and a novel idea of time limiting. The scheme is linearized as much as possible using first order accurate predictors for Runge-Kutta stages and the final scheme contains only the nonlinearity of the flux function: in particular it can be applied by solving only linear systems for linear equations. The limiting in time is needed to control spurious oscillations arising from the fact that waves can cross more than one computational cell in each timestep.

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High-Order Shock-Capturing Schemes for Non-Convex Special Relativistic Hydrodynamics

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Keywords: high order numerical methods, special relativistic hydrodynamics

ABSTRACT

In order to simulate complex dynamics in special relativistic hydrodynamics (SRHD) in which phase changes appear (as in high-energy astrophysical events like coalescence of binaries of black holes) it is required that thermodynamics be represented by non-convex equations of state [1]. In non-convex dynamics, in addition to elementary waves such as rarefactions and shocks, combinations of these can appear developing complex structures.

We study the behavior of the high order version of two widely used shock capturing numerical schemes for relativistic hydrodynamics [3, 5, 6] when closing the SRHD system of equations with the phenomenological non-convex equation of state proposed by Ibáñez et al, [4]. We analyze their behaviour for different prescribed viscosities comparing their approximation against the exact solution of Riemann problems developing non-convex dynamics presented in [2]. In particular we examine the accuracy of the numerical approximation of the methods in the density shell of two blast wave problems where composite waves are formed. We also evaluate the stability for highly relativistic colliding slabs problems.

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Discontinuous Galerkin simulations for continuum-rarefied gas flows over aerospike blunt body based on regularized 13-moment model

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Keywords: DG method, R13-moment model, aerospike blunt body, continuum-rarefied flows.

ABSTRACT

Numerical investigations of high speed flows have long been an interesting and difficult topic in gas dynamics. During their travel through the environment, high-speed vehicles encounter considerable aerodynamic drag and aerothermal heating [1]. The blunt body configuration is currently in use to reduce aerodynamic heating to the surface but it leads to significant increase in the drag. For re-entry vehicles, which need a high drag coefficient with little aerodynamic heating, blunt body design is effective; nevertheless, it is not recommended for other high-speed vehicle uses. Aerospike ahead of the blunt body in high-speed vehicles is one technique to reduce the drag and aerodynamic heating [2]. The degree of rarefaction, often known as the Knudsen number, is used to characterize the high-speed flow regimes. When the value of Knudsen number increases, the gas flow regime becomes rarefied, and subsequently, it turns into the free molecular regime. Without proper velocity slip and temperature jump boundary conditions, the classical Navier-Stokes-Fourier (NSF) equations may not be able to accurately forecast the drag and aerodynamic heating operating on these vehicles.

Therefore, to overcome these shortcomings, the Boltzmann transport equation-based gas kinetic models are frequently considered. Recently, Torrilhon and Struchtrup introduced the regularized 13-moment (R13) model based on the asymptotic expansion of the Boltzmann equation [3, 4]. This model describes all known rarefaction effects, like Knudsen layers, Knudsen minimum, heat flux without temperature gradient etc., with good accuracy. In this work, R13-moment model in conjunction with Maxwell slip and Smoluchowski jump conditions are considered to simulate the continuum-rarefied gas flow over aerospike blunt body. For this purpose, an explicit mixed modal discontinuous Galerkin method based on triangular meshes is developed for solving two-dimensional R13-moment model. The effects of aerospike for various ratios of spike length (L) to the diameter (D) of the blunt body are investigated a wide spectrum of rarefied hypersonic gas flows. Numerical simulations reveal that the L/D ratio substantially affect the drag, aerodynamic heating, and stability of the high-speed vehicles.

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A class of high order ADER-DG schemes for the two dimensional nonlinear advection-diffusion equation

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Keywords: high order numerical methods, Discontinuous Galerkin schemes, advection-diffusion.

ABSTRACT

In this talk, we present arbitrary high order space-time schemes to solve the unsteady **advection-diffusion equation** in 2D. We propose to combine Discontinuous Galerkin (DG) methods in space to an ADER (Arbitrary high order DERivatives) technique in time. This association allows to perform the time integration at the same order of accuracy as the one provided by the DG discretization in space.

In its most recent form, the **ADER-DG scheme** is composed of two successive stages: prediction and correction [1]. In the **prediction** step, we determine a local space-time Galerkin predictor, that is to say in each space-time cell, in order to obtain a first estimation of the solution. The **correction** step then allows to recombine the information between neighbouring cells through numerical fluxes and to impose the boundary conditions.

It is well known that using $\mathcal{P}_k$ polynomials to solve the pure advection equation always provides a DG scheme of order $(k+1)$. However, this order of precision is limited to k in some cases for even values of k for the pure diffusion problem [2]. We will thus present three versions of **interior penalty** (IPDG) and a Cattaneo **relaxation** technique to handle the diffusion term. The comparison of the IPDG approaches has already shown that the symmetric variant is preferable [3], but we will provide a deeper analysis by confronting them to the Cattaneo method.

The 1D linear case has been detailed in [3]. This presentation will focus on extending these results to **2D test cases** on cartesian grids. We will show that arbitrary high order ADER-DG schemes are competitive regarding the computational cost. They also offer interesting perspectives to solve the incompressible Navier-Stokes equations without resorting to Chorin-Temam methods.

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Slip Wall BC in Curved Domains for FE ALE Hydrodynamics

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Keywords: high order finite elements, ALE methods, wall BC.

ABSTRACT

We present a method that enables ALE simulations with free-slip wall boundary conditions in complex domains with curved boundaries. This work expands upon the high-order finite element ALE approach [1], which, prior to this study, always utilized free-slip boundaries aligned with the coordinate axes. In the Lagrangian phase, the variation formulation of [2] is enhanced by Nitsche-type terms to enforce the BC weakly [3]. The new terms preserve the structure of the original discretization, i.e., all mass matrices are constant and total energy is conserved. In the mesh optimization phase, we extend the original Target-Matrix Optimization Paradigm formulation [4] with a penalty term that acts on the boundary nodes. This penalty enforces tangential mesh relaxation while taking into account the discrepancies between the analytic curves and the boundary node positions that are obtained from the Lagrangian motion. The remap phase does not require any modifications as the original derivation of the pseudotime-advection formulation agrees with the tangential mesh motion at the boundaries. The robustness and accuracy of the method are validated on a set of Sedov-type ALE tests involving nontrivial curved boundaries.

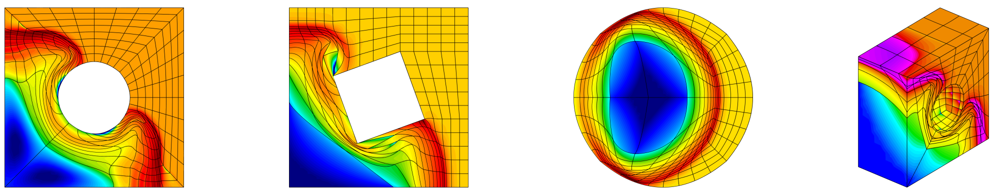

Figure 1: Examples from [3] of Lagrangian simulations on curved meshes with slip wall BC.

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Divergence–free preserving schemes: how to fix stabilization terms in continuous Galerkin

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Keywords: divergence-free, continuous Galerkin, SUPG, arbitrary high order numerical methods.

ABSTRACT

The emergence of physical structures and equilibrium solutions, such as divergence-free solutions in contexts like shallow water and magneto-hydrodynamics, poses a significant challenge. A simple linear approximation of such systems that already show these behavior is the linear acoustic system of equations. We focus on Cartesian grid discretizations of such systems in 2 dimensions and in the preservation of stationary solutions that arise due to a truly multidimensional balance of terms, which corresponds to the divergence-free solutions for acoustic systems. Conventional methods, like the continuous Finite Element SUPG, face limitations in maintaining these structures due to the employed stabilization techniques that do not effectively vanish when the discrete divergence is zero. We propose to use the Global Flux procedure, which has proven to be successful in preserving 1-dimensional equilibria [1, 2], to define some auxiliary variables guiding a suitable discretization of both the divergence and stabilization operators [3]. This approach naturally preserves divergence-free solutions and more intricate equilibria involving various sources. Moreover, this strategy facilitates the identification of discrete equilibria of the scheme that verify boundary or initial conditions. We use the Deferred Correction time discretization, obtaining explicit arbitrarily high order methods. Numerous numerical tests validate the accuracy of our proposed scheme compared to classical approaches. Our method not only (discretely) preserves divergence-free solutions and their perturbations but it also maintains the original order of accuracy on smooth solutions.

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*ADDA*ptive Numerical Framework for iLES of Compressible Flows

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Keywords: Turbulence, Compressible Flow, High-Order Methods, CWENO.

ABSTRACT

Advancements in high-resolution, high-order numerical schemes have enabled the wide-spread popularity of iLES of turbulent flows, since these schemes can effectively simulate the effects of unresolved flow scales through their numerical dissipation and dispersion errors, often acting as subgrid-scale models. However, managing the intertwined numerical dissipation and dispersion to achieve physically meaningful results is challenging, in particular for under-resolved grid settings due to the coexistence of discontinuities and smooth flow features. This work presents an adaptive dissipation/dispersion adjustment (ADDA) algorithm that identifies well-resolved and under-resolved regions and modulates the numerical dissipation/dispersion of high-order non-linear schemes, while also acting on the flux-dissipation terms of suitable selected Riemann solvers. The new ADDA algorithm implemented in the open-source UCNS3D CFD solver is tested across a wide-range of compressible flow problems.

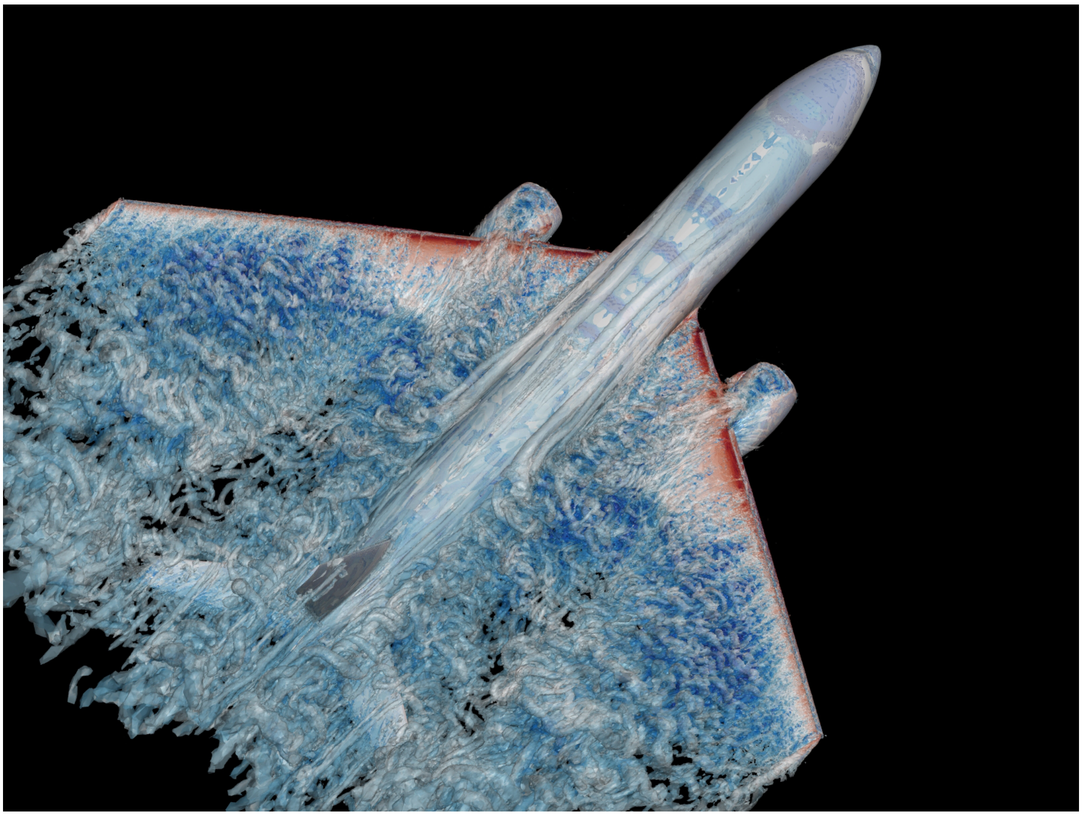

Figure 1: Instantaneous iso-surfaces of Q-criterion coloured by Mach number for the turbulent flow past the NASA-CRM High-Lift Configuration, obtained with the present ADDA-CWENOZ4 method using UCNS3D.

Strongly consistent low-dissipation WENO schemes for finite elements

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Keywords: hyperbolic conservation laws, CG and DG methods, WENO scheme, residual-based nonlinear weights, consistency, a priori estimates.

ABSTRACT

We propose a way to maintain strong consistency and facilitate error analysis in the context of dissipation-based WENO stabilization for continuous and discontinuous Galerkin discretizations of conservation laws. Following Kuzmin and Vedral (J. Comput. Phys. 487:112153, 2023) and Vedral (arXiv preprint arXiv:2309.12019), we use WENO shock detectors to determine appropriate amounts of low-order artificial viscosity. A key novelty of our approach lies in the use of residual-based weights for candidate polynomials of a WENO reconstruction. The shock-capturing terms of our stabilized Galerkin methods vanish if residuals do. This enables us to preserve the Galerkin orthogonality property and achieve improved accuracy compared to weakly consistent alternatives. As we show in the context of convection-diffusion-reaction (CDR) equations, nonlinear local projection stabilization terms can be included in a way that preserves the coercivity of local bilinear forms. For the corresponding Galerkin-WENO discretization of a CDR problem, we rigorously derive a priori error estimates. Additionally, we demonstrate the stability and accuracy of the proposed method through one- and two-dimensional numerical experiments for hyperbolic conservation laws and systems thereof. The numerical results for representative test problems are superior to those obtained with traditional WENO schemes, particularly in scenarios involving shocks and steep gradients.

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Challenges for Physics-Based Model Order Reduction in Data Assimilation

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Keywords: model order reduction, parametrized partial differential equations, data assimilation, inverse problems.

ABSTRACT

In this talk, we will discuss some recent developments as well as opportunities and challenges in the development and use of physics-based model order reduction (MOR) for parametrized partial differential equations in the context of data assimilation.

Data assimilation methods seek to merge observational data with physical models in a statistically consistent way, accounting for uncertainties in both model predictions and observations. Data assimilation has been widely used in meteorology, but there is growing interest in its application to complex engineering systems. However, the main computational bottleneck in data assimilation (as with most uncertainty quantification tasks) is the solution of the forward PDE problem for many instantiations of the system parameters. We thus show how parametric reduced order models can be gainfully used to address this high computational cost. We focus particularly on the development and use of reduced models in ensemble Kalman inversion (EnKI) and ensemble Kalman filtering (EnKF). First, we show how the EnKI algorithm can be adjusted to mitigate the impact of the model bias introduced by the use of the reduced order model. Second, we consider the use of adaptive reduced order models in multi-fidelity and multi-level EnKF techniques. Here, we combine information extracted from a limited number of full order solutions and information from inexpensive reduced models, thus permitting accurate predictions at lower computational cost. If time permits, we will also briefly consider further challenges such as multi-scale aspects and high-dimensional parameters, as well as other uncertainty quantification tasks such as optimal sensor placement.

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Monolithic local subcell DG/FV convex property preserving scheme: Is entropy stability really needed?

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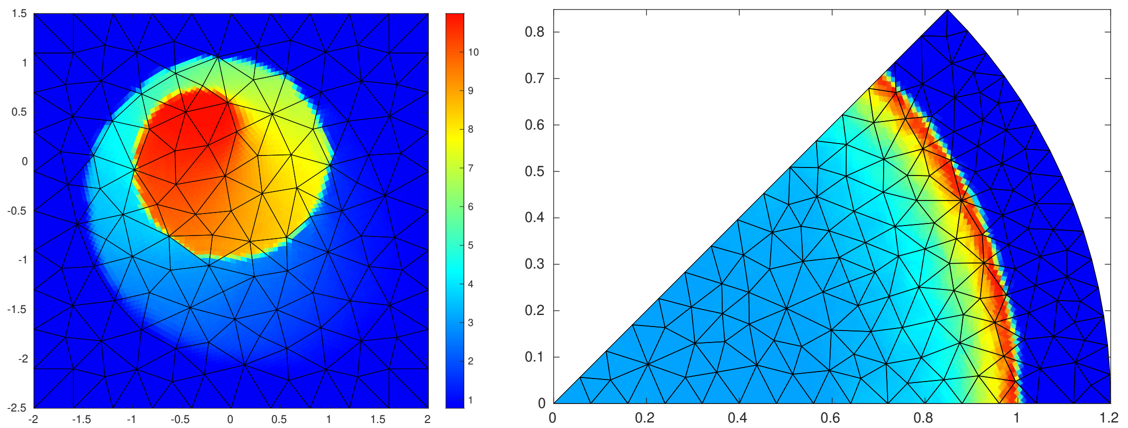
Keywords: high-order scheme, entropy stability, structure preserving schemes, hyperbolic PDEs.

ABSTRACT

This talk aims at presenting a subcell monolithic DG/FV convex property preserving scheme solving system of conservation laws on 2D unstructured grids. This is known that discontinuous Galerkin (DG) method needs some sort of nonlinear limiting to avoid spurious oscillations or nonlinear instabilities which may lead to the crash of the code. The main idea motivating the present work is to improve the robustness of DG schemes, while preserving as much as possible its high accuracy and very precise subcell resolution. To do so, a convex blending of high-order DG and fist-order finite volume (FV) scheme will be locally performed at the subcell scale where it is needed. To this end, by means of the theory developped in [1, 2], we first show that it is possible to rewrite DG scheme as a subcell FV scheme on a subgrid provided with some specific numerical fluxes referred to as DG reconstructed fluxes. Then, the monolithic DG/FV scheme will be defined as following: to each face of each subcell will be assigned two fluxes, a 1st-order FV one and a high-order reconstructed one, that will be in the end blended in a convex way. The goal is now to determine, through analysis, optimal blending coefficients to achieve the desire properties. Numerical results on various type problems will be presented to assess the very good performance of the design method, see Figure 1.

By means of this subcell monolithic framework, we will also attempt to address the following questions: what do we mean by entropy stability? What is the cost of such constraints? Is this absolutely needed? Pieces of answers will be given.



(a) 8th-order on 242 cells (b) 6th-order on 271 cells

Figure 1: Numerical solutions for respectively the rotating KPP and Euler Sedov problems.

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P-Anisotropic H-Isotropic adaptive discontinuous Galerkin methods for turbulent flows

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Keywords: High-order discontinuous Galerkin method, anisotropic p-adaptation, least-squares reconstruction, adaptive mesh refinement

ABSTRACT

High-Reynolds number flows feature highly anisotropic regions. To allow for adaptive methods that can efficiently resolve these regions, anisotropic strategies are needed. Furthermore, previous results have shown that purely p-adaptive methods as well as purely h-adaptive methods have severe shortcomings in their applicability and that combined hp-adaptive methods offer superior performance. In this work, an h-adaptive framework using isotropic mesh subdivision is utilized, the anisotropic adaptation is handled by the anisotropic p-enrichement. For this, a modal DG basis in physical space is used [1]. To allow for anisotropic adaptation, a subspace of the full polynomial space is chosen individually for each element. To select candidate elements for refinement a residual-based indicator [2] is used. The anisotropic p-adaptivity is driven by a least-squares reconstruction based competitive refinement technique. The combination of this anisotropic p-adaptivity with h-adaptivity provides better accuracy per degree of freedom compared to a second order h-adaptive method as well as an isotropic hp-adaptive method. This can be seen in Fig. 1 for the example of the turbulent flat plate. The work was carried out in the CFD software by ONERA, DLR and Airbus (CODA).

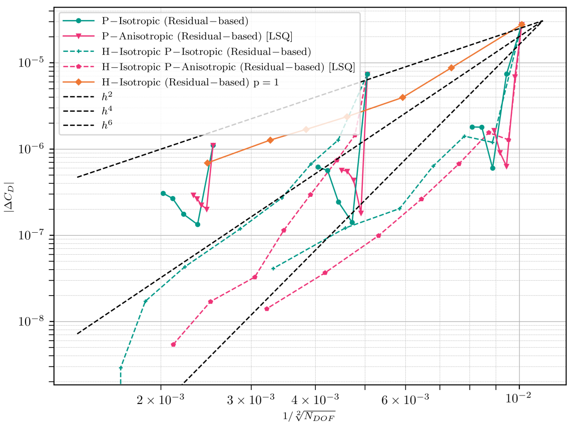

Figure 1: Convergence of the drag coefficient error for the turbulent zero-pressure gradient flat plate.

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Compatible finite element discretization of the time-dependent magnetic advection-diffusion equation with application to magnetohydrodynamics

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Keywords: high order numerical methods, structure preserving schemes, compatible finite element methods, magnetohydrodynamics.

ABSTRACT

Compatible finite elements methods are widely employed for solving vector Laplace-like problems (e.g. Maxwell, Stokes, linear elasticity, . . .), since they have many structure-preserving properties. However, comparatively less attention has been directed towards convection-like problems (incompressible Euler equations, ideal magnetohydrodynamics, . . .). In this presentation, the focus lies on addressing the magnetic advection-diffusion equation, introducing a novel approach building upon the interpolation-contraction method proposed by Hiptmair and Pagliantini [1], based on an approximate Rusanov-like multi-dimensional Riemann solver. Notably, we employ "small simplices" [2] for the interpolation, simplifying the associated curl operator matrix to an incidence matrix. Regarding the time discretization, we adopt a Lax-Wendroff strategy as an alternative to the classical method of lines. The resulting method exhibits stability, high-order accuracy, and preserves the divergence-free nature of the magnetic field without necessitating matrix inversions for purely convective scenarios. To mitigate the Gibbs phenomenon, we incorporate an a posteriori MOOD-style artificial resistivity guided by a discrete maximum principle. Finally, we integrate the aforementioned method with a staggered semi-implicit hybrid finite volume/finite element solver for the incompressible Navier-Stokes equations, obtaining a novel scheme for incompressible magnetohydrodynamics [3].

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Approximation of Moment Equations for Modeling Sedimentation in Suspensions of Rod-Like Particles

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Keywords: High dimensional problem, Kinetic Equation, Moment Equations.

ABSTRACT

We focus on a model (Helzel & Tzavaras [1]), which defines a sedimentation process in dilute suspension of rigid rod-like particles under the influence of gravity. This model is defined by a system of partial differential equations which couples a kinetic Smoluchowski equation to a macroscopic flow equation.

The Smochluchowski equation is a drift-diffusion equation on the sphere, which needs to be solved at every point of the flow domain. Our goal is to construct an efficient numerical method for this high-dimensional problem (i.e., a 5-dimensional problem plus time). This will be done by replacing the high-dimensional Smoluchowski equation by a lower-dimensional system of moment equations. The moment equations form a hyperbolic system with source term that can be approximated with standard numerical methods such as LeVeque's wave propagation algorithm ([3]). This is an extension of previous work ([2]), where for simplicity the Smoluchowski equation was restricted to S^1 instead of S^2. We will present current work in progress for the full model. In particular we will present a spectral method for the Smoluchowski equation with externally imposed velocity gradient that simulates the evolution described by the source term of the hyperbolic moment system.

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A semi-implicit finite volume scheme for fluid-structure interaction problems

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Keywords: Fluid-structure interaction, Semi-implicit finite-volume scheme, Subgrid, Nested Newton, Shallow water

ABSTRACT

In this poster, the key concepts of a novel semi-implicit finite volume scheme (SIFSI) are presented for the coupled solution of the water flow and the movement of one or more floating structures. The model is based on the hydrostatic pressure assumption and the shallow water equations. The coupling is achieved via a nonlinear volume function in the mass conservation equation that depends on the coordinates of the floating structures. Furthermore, the nonlinear volume function allows for the simultaneous existence of wet, dry and pressurized cells in the computational domain. The resulting mildly nonlinear pressure system is solved using a nested Newton method. The accuracy of the volume computation is improved by using a subgrid. At each time step, the position of the floating objects is computed by solving a set of ordinary differential equations for their six degrees of freedom. For a set of test cases, the model has been applied and compared with available exact solutions to verify the correctness and accuracy of the proposed algorithm.

The model is able to treat fluid-structure interaction in the context of geophysical free surface flows in an efficient and flexible way, and the employed nested Newton method rapidly converges to a solution.

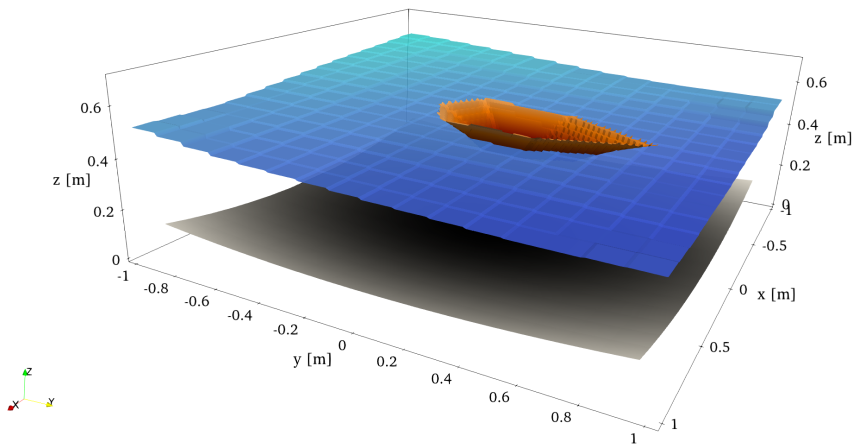

Figure 1: Simulation of a ship floating in a lake.

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Solving Incompressible Navier-Stokes with High-Order Mimetic Methods

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Keywords: high order numerical methods, parallel implementation, mimetic methods.

ABSTRACT

Title: Exploring High-Order Mimetic Methods for Navier-Stokes Equations on Curvilinear Grids: Implementation and Testing

Abstract: This talk delves into high-order mimetic methods for solving the Navier-Stokes equations on curvilinear grids. Mimetic methods, renowned for preserving key properties of the underlying mathematical operators [1, 2], offer a promising avenue for accurately capturing fluid flow phenomena. Using mimetic methods to solve the Navier-Stokes equations on curvilinear grids successfully models complex flows on curvilinear grids[3]. Our focus lies particularly on the implementation and testing of high-order mimetic schemes on curvilinear grids within a parallel computing framework [4] .

Existing codes boast parallel capabilities and the flexibility to seamlessly transition between low and high-order methods [5]. However, the high-order variants remain untested, both in terms of accuracy and computational efficiency, especially on curvilinear grids. We propose to address this gap by implementing high-order mimetic schemes alongside high-order time integration methods, evaluating their performance and accuracy on curvilinear grids.

By conducting systematic tests and comparisons, we aim to assess the efficacy of high-order mimetic methods in accurately simulating fluid flow dynamics, particularly on complex curvilinear grids. The talk will highlight the implementation strategies, numerical considerations, and anticipated challenges in testing the proposed schemes' spatial and temporal accuracy.

Through this investigation, we seek to advance the understanding of high-order mimetic methods for Navier-Stokes equations and provide insights into their potential applicability in large real-world fluid dynamics simulations.

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Active Flux Methods for Hyperbolic Systems using the Method of Bicharacteristics

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Keywords: High order numerical methods, Cartesian Grid Active Flux Method, Method of Bicharacteristics.

ABSTRACT

The Active Flux method is a third order accurate finite volume method for hyperbolic conservation laws, which in contrast to classical finite volume methods not only uses cell cell average values but also point values of the conserved quantities. The resulting method is truly multi-dimensional, fully discrete and has a compact stencil in space and time. A crucial component of Active Flux methods is the evolution formula for the update of the point values. For that matter we discuss the method of bicharacteristics, which provides a general framework for the derivation of truly multidimensional approximative evolution operators for linear systems. We then apply the method of bicharacteristics to the linearised Euler equations to obtain a Cartesian grid Active Flux method. For the nonlinear Euler equations we use the same evolution operator linearising those equations around each point value combined with an iterative approach.

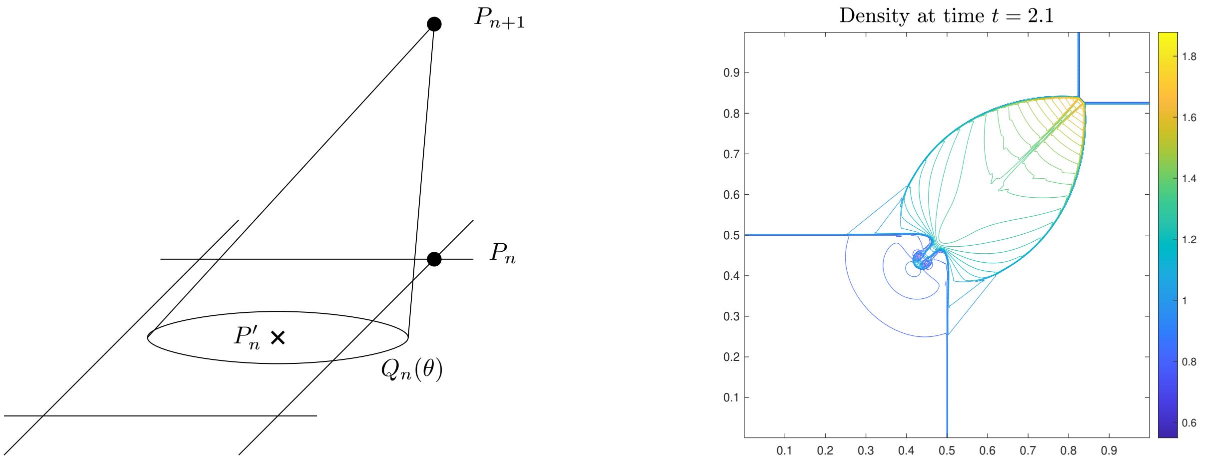

Figure 1: Update of a corner point value using the method of bicharacteristics (left). Solution to the Euler equations with a classical Riemann problem as initial data (right).

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SYNChronised numerical methods

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Keywords: structure preserving schemes.

ABSTRACT

Discretising partial differential equations (PDEs) can produce numerical errors [11], and in particular symmetry errors. Traditionally, symmetry is *fitted* into the numerical method [5] based on three approaches. The first is to align the mesh with the underlying symmetry and adapt the numerical method accordingly [3, 9]. A second approach is to use the natural coordinate frame and to use a coordinate transformation from physical to computational space. For this type of strategy the particular symmetry is contained within the Jacobian for the transformation and is embedded in a Geometric Conservation Law [10]. Finally, a third approach is to modify the numerical method to reduce the asymmetric errors, inherent within the truncation terms, by using more information from the wider computational mesh stencil [8]. Partial differential equations remain unchanged when certain transformations are performed; these *symmetries* leave the equations invariant after a given transformation. Lie group theory has been shown to be a general way of determining these symmetries [2, 7] and has been used to find analytic solutions to many partial differential equations, and in particular, hydrodynamic related equations [4, 1]. However, in general the discretised equations do not preserve these symmetries, unless specially constructed [6]. By extending Lie groups to the discrete system we will develop the governing principles of SYNChronised numerical methods; SYmmetries of the partial differential equations are Numerically Captured. By introducing the concept of numerical symmetry breaking, we will show why some numerical techniques are more suitable than others for a given application.

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Introducing RESCUER: Resilient Solutions for Coastal, Urban, Estuarine and Riverine Environments

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Keywords: coastal modeling, river and estuary dynamics, urban flooding, water quality

ABSTRACT

Coastal areas globally, including Europe, face mounting pressures from population growth, rising standards of living, and climate change-induced sea level rise. These stressors increase vulnerabilities such as microplastic pollution, saltwater contamination, chemical spills, and flood risks. Consequences include damage to structures, infrastructure, and tourism revenue, as well as broader societal impacts like poor water quality, biodiversity loss, and crop failures. RESCUER is a MSCA Doctoral Network funded by the European Commission Pillar 1 Excellence, aiming to educate the next generation of researchers combining numerical and modeling experience with and understanding of physical and biological processes, and a keen awareness of the challenges faced by us due to the climate change and subsequent changes in the risk profile of coastal communities. RESCUER integrates four research themes: **WP1: Coastal modeling and hazards** – focusing on phase-resolved wave forecasting tools and the study of wave processes and hazards related to infragravity waves and waves propagating over steep bed topography; **WP2: River and estuary dynamics** – focusing on flow and sediment transport in the river/estuary contact area, and to study sediment transport over saturated subsurface flow; **WP3: Urban flooding –** focusing on the construction of higher oredr models which include floating debris; **WP4: Water quality** – focusing on the spreading of pollutants and the kinetics of decomposition. In order to address these phenomena, we will leverage a number of novel methods developed in the last few years such as Graphical Processing Unit techniques, new efficient methods for boundary conditions and entropy stability [1] as well as higher-order accurate models for wave generation and propagation. When combined, these techniques will enable us to model the impact of phenomena such as infragravity waves and the influence of steep bathymetry with high accuracy, and also allow the establishment of real-time operational wave forecasting in coastal locations. For river and estuary dynamics, we will focus on coupling hydrodynamic codes with transport/diffusion models which need to interact dynamically in order to provide a complete picture of current and wave conditions as well as spreading of contaminants. For urban flooding, these challenges will be addressed by the utilization of a shallow-water model and the use of high-order DG methods on unstructured grids.

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A positivity preserving Active Flux method for the Vlasov-Poisson System

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Keywords: Active Flux, positivity preserving schemes, Vlasov equation, plasma physics.

ABSTRACT

In this work we present a positivity-preserving third-order accurate Active Flux method for the 1+1-dimensional Vlasov-Poisson system. The Vlasov-Poisson system, a simplification of the Vlasov-Maxwell system, models the evolution of a collisionless plasma [4]. Active flux methods [1] are finite volume methods that use, in addition to cell averages, point values as degrees of freedom. In the presented work, the point values are evolved in time using the characteristic form of the equation, while cell averages are updated using the conservative form of the equation. The availability of additional point values allows us to compute moments of the Vlasov equation more easily, which enables the construction of an efficient third-order accurate method for the Vlasov-Poisson system. In order to guarantee the non-negativity of the approximated solution, a positivity-preserving flux limiter [2] will be introduced which extends our previous method from [3].

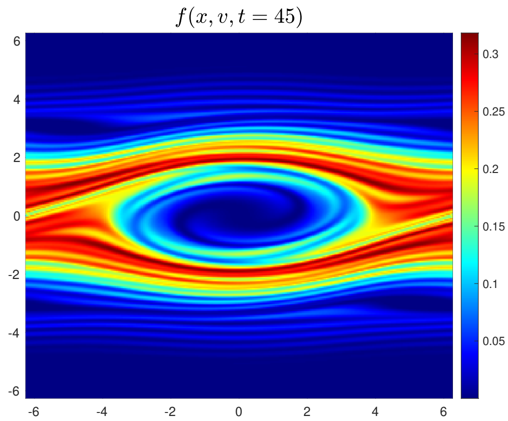

Figure 1: Solution of the 2-Stream Instability problem at time $t = 45$.

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Improvements of the 3D Lagrangian Lax-Wendroff scheme with artificial dissipation

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Keywords: Lagrangian methods, ALE methods, Unstructured meshes

ABSTRACT

We have developed a three-dimensional formulation [1] of the Lagrangian Lax-Wendroff scheme with artificial dissipation [2]. This method does not rely on nodal solvers, unlike many modern cell-centered methods, but uses the predictor-corrector formalism to obtain the velcocity of mesh nodes directly. The scheme can be formulated so that it satisfies GCL either approximately or exactly. The shock waves are treated by additional artificial viscosity and energy dissipation terms. Such formulation is straightforward and leads to a simple and efficient code.

We will discuss the applications of this method to inviscid compressible flows on unstructured meshes, hyperelastic solids (2D applications with linear elasticity were already investigated [3]) and a possible extension of the framework to direct Arbitrary Lagrangian-Eulerian computations. The performance of the proposed method is shown on 3D numerical benchmarks, showing low mesh imprinting and spherical symmetry preservation on non-conforming meshes. The artificial dissipative terms help to reduce the wall heating effect and unwanted vorticity generation on non-aligned shock fronts (Fig. 1).

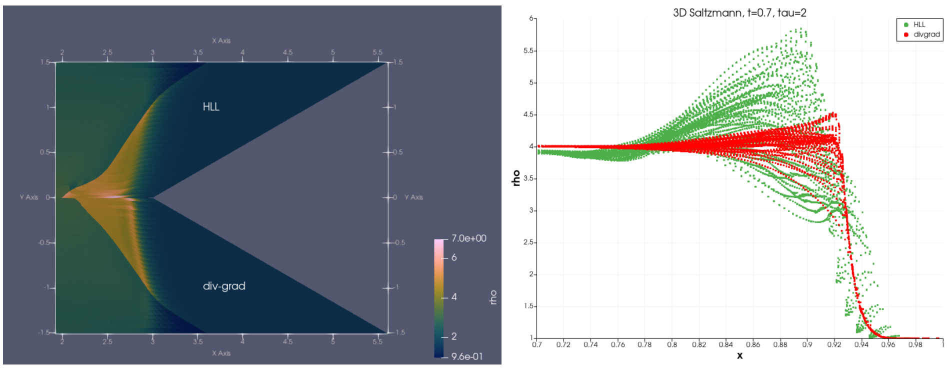

Figure 1: Comparison of different artificial dissipation formulations in the LW+ scheme – shock refraction over skewed interface (left); density scatter plot of the 3D Saltzmann test (right).

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High order ImEx method for the shallow water model

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Keywords: shallow water equations, implicit explicit schemes, high order in time, low Froude number.

ABSTRACT

This poster is devoted to the development and analysis of a robust and efficient high order numerical scheme for the shallow water flows in the low Froude number limit. We focus on ocean and coastal simulations at different scales, in particular, on the variation of the Froude number that goes from 1 at the coastline to two or three orders less offshore. Due to the great ocean's depth, classical hyperbolic schemes like Riemann solvers are not efficient [1]. In order to propose an efficient method in such regime, a part of the system has to be considered implicitly, leading to an ImEx (Implicit Explicit) scheme. In order to limit the size and number of linear systems to be solved, the CPR scheme [2] is a good first order candidate. The CPR approach is a fully diagonal segregated method which only relies on the implicit treatment of the water height and hybrid mass fluxes using explicit velocities. The method allows to avoid resolution of large linear systems. Concerning the high order in time integration, several Runge-Kutta schemes can be found in the literature [3] in the context of ImEx schemes, however to limit the number of linear systems to solve, we focus on Cranck Nicolson schemes. For the space discretization, a classical second order MUSCL reconstruction is used. We finally show, thanks to one- and two-dimensional test cases, that the developed scheme achieves the theoretical second-order rate of convergence. Furthermore, we conduct a comparative analysis of CPU times between the ImEx and explicit schemes, revealing important computational savings with the ImEx scheme particularly under the low Froude regime.

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A Multi-Threaded High-Order Lagrangian Scheme

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Keywords: High-Order numerical methods, ALE, Lagrangian methods.

ABSTRACT

Conventional staggered-grid methods for hydrodynamics (SGH) normally use low-order finite elements that have straight edges which do not allow the mesh to follow highly disturbed flows particularly well. High-order methods use elements that have curved edges which do allow the mesh to follow highly disturbed flows. Unlike SGH schemes which implement thermodynamic quantities such as pressure, density and internal energy as cell-centred or edge-centred variables, high-order schemes implement these quantities as functions across elements. Artificial viscosity tensor implementations benefit from this and significant improvements in symmetry on meshes that are initially non-uniform are obtained [1,3]. High-order schemes may require less ALE allowing more Lagrangian mesh motion and reduced numerical dissipation as they require fewer ad-hoc fix-ups. These properties motivate us to make some assessments of the use of high-order Lagrangian hydrodynamics in some of our applications. We illustrate the current stage of our high-order implementation which is now operational in axisymmetric (rz) geometry and has a material strength model [2]. We demonstrate this on various standard hydrodynamic test cases including Noh, Sedov and Saltzmann. These tests indicate symmetry, robustness and fidelity improve when using a high-order compared to an SGH scheme.

We are also interested in how to extract concurrency via threading when deploying high-order schemes on parallel computers. Our implementation has been threaded in a number of areas of the code, via OpenMP. OpenMP has been chosen as a first step towards accelerating our hydro kernels on GPU hardware because the code changes needed to utilise it effectively improve thread-safety. We show some initial gains in performance and runtime reduction we have achieved using this approach in areas of the code that include force assembly, quadrature updates and the internal energy update. The force assembly is extremely intensive computationally, and on a single compute resource usually dominates run-time. This part of the code is therefore a prime target for optimisation and was thus selected for threading; any improvement in performance here promises a large reduction in run-time. The quadrature update is a second area of the code we selected for threading. This is called frequently on each numerical quadrature point, and even coarse meshes in our high-order code contain a significant number of quadrature points. Our future plans are to extend the threading to more kernels and to achieve thread-safety in all parts of the code. This will allow us to deploy our high-order code efficiently on GPU hardware.

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Recent Advances in the Target-Matrix Optimization Paradigm for High-Order Mesh Adaptivity

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Keywords: hrp-adaptivity, high-order finite elements, mesh optimization.

ABSTRACT

The Target-Matrix Optimization Paradigm (TMOP) is a technique for adapting high-order meshes. In this method, optimized mesh nodal positions are determined by minimizing a functional that depends on each element's *current* and *target* geometric parameters: size, aspect-ratio, skew, and orientation. In previous work, we have demonstrated use of TMOP for solution-driven adaptivity in two- and three-dimensional problems [1, 2, 3].

Recently we have augmented the TMOP-based functional with a penalization term that depends on a subset of mesh nodes aligning with a target surface prescribed as the zero of a level-set function. Minimizing this functional for a given mesh and a level-set thus results in a body-fitted mesh with good element quality [4]. This technique has proven to be effective for automating meshing in topology optimization featuring complex curvilinear geometries. We have also augmented this approach with p-adaptivity to produce mixed-order meshes that use high-order elements only near curved surfaces [5]. For cases where rp-adaptivity alone is not sufficient for desired accuracy, we use adaptive mesh refinement to formulate an hrp-adaptivity approach. The resulting method is demonstrated to be robust at obtaining mixed-order body-fitted meshes that are computationally efficient than uniform meshes, and extends to different element types (quadrilaterals/triangles/tetrahedron/hexahedra).

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